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A high-order cnoidal wave theory

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A method is outlined by which high-order solutions are obtained for steadily progressing shallow water waves. It is shown that a suitable expansion parameter for these cnoidal wave solutions is the dimensionless wave height divided by the parameter m of the cn functions: this explicitly shows the limitation of the theory to waves in relatively shallow water. The corresponding deep water limitation for Stokes waves is analysed and a modified expansion parameter suggested.

Cnoidal wave solutions to fifth order are given so that a steady wave problem with known water depth, wave height and wave period or length may be solved to give expressions for the wave profile and fluid velocities, as well as integral quantities such as wave power and radiation stress. These series solutions seem to exhibit asymptotic behaviour such that there is no gain in including terms beyond fifth order. Results from the present theory are compared with exact numerical results and with experiment. It is concluded that the fifth-order cnoidal theory should be used in preference to fifth-order Stokes wave theory for wavelengths greater than eight times the water depth, when it gives quite accurate results.

1. Introduction

A knowledge of the flow field due to the passage of water waves has become increasingly important as more structures are built to resist loads caused primarily by wave-induced fluid motion in hostile marine environments. Often, the waves are so long in relation to the water depth that existing theories are no longer adequate. In the absence of an accurate shallow water theory, however, theories which are best suited to deep water continue to be applied, notably the fifth-order Stokes wave solution (De 1955; Skjelbreia & Hendrickson 1961).

Several high-order approximations to irrotational water waves of constant form have appeared in recent years, often based on Fourier series, but where convergence is slow, if at all, for shallow water. These solutions are often numerical, and of an inverse formulation, and are generally of such high order, that it is difficult to obtain expressions for physical quantities as functions of position for practical use. The presentation of results has been limited to tables of integral quantities for a range of wave lengths and heights. However, these methods have achieved real success in obtaining numerically exact solutions for the first time (Schwartz 1974; Cokelet 1977). A survey and comparison of the methods is given in Cokelet's paper.

The first shallow-water theory of periodic waves was given by Korteweg & de Vries (1895), who showed that the first approximation to the surface profile of steadily progressing waves in shallow water was cnoidal. Littman (1957) proved the existence of such solutions for sufficiently small waves. Laitone (1960) obtained the second-order approximation to these cnoidal waves. A high-order solution was attempted by Monkmeyer (1970), who assumed a cnoidal type of solution, obtaining coefficients of this solution to fifth order in wave height. Unfortunately, this was rather difficult to apply to a practical problem, as he used an inverse method, obtaining equations for cartesian co-ordinates in terms of the velocity potential and stream function. Also, the equations were solved numerically so that results could be presented only in the form of tables of coefficients for different wavelengths. Finally, in solving the equations, the elliptic cn functions were expanded as Fourier series, inhibiting their usefulness in shallow water.

In view of the lack of an accurate theory for shallow water waves, it was decided to produce such a theory, but one which contained the following features.

(i) Dependence on water depth and wave height would be included specifically so that no equations need be solved numerically in any subsequent application. Rather, any calculations would be limited to the evaluation of series.

(ii) That it would be direct, giving quantities as functions of time and position rather than of stream function and potential. Thus the theory was to be basically a high-order extension of Laitone's second-order cnoidal wave solution (Laitone 1960, 1965), a shallow water theory complementary to the fifth-order Stokes wave theory of Skjelbreia & Hendrickson (1961).

Using a Rayleigh–Boussinesq series previously applied to solitary waves (Fenton 1972), exact equations are set up in §2, into which series expressions are substituted in §3, leading to the formation of a recursion relationship for a solution of any order. Computer programs were written to perform the extremely long manipulations and to obtain the solutions. In §4 it is shown that all quantities of the cnoidal wave solution are more properly given by series in ϵ/m, where ϵ is the dimensionless wave height and m is the parameter of the Jacobian elliptic functions introduced. The use of this quantity as an expansion parameter explicitly shows the applicability of the method to longer waves ($m = 1$ for solitary waves but becomes smaller for shorter waves, making the expansion quantity ϵ/m larger).

Solutions were obtained to ninth order, however it was subsequently found that there was no justification in going beyond the fifth order. All results to this order are presented as tables of coefficients in series expansions. Section 4.3 shows how a practical problem involving known water depth, wave height and wavelength or period can be solved to give ϵ and m, which can then be used in the series of §4.4 for the wave profile, and fluid velocities, accelerations and pressures. In §4.5 a number of series for integral properties of the wave train are given, such as wave impulse, energy, radiation stress, wave power, and mean Stokes drift velocity.

Results from the present theory are compared with previous work in §5. The breakdown of the Stokes wave approach in shallow water is analysed and compared with the breakdown of cnoidal wave approximations in deep water. Wave speed is then used as the criterion for comparing the present work with Stokes wave theories. Finally, fluid velocity profiles given by the present theory are compared with Stokes wave profiles and with experimental results.

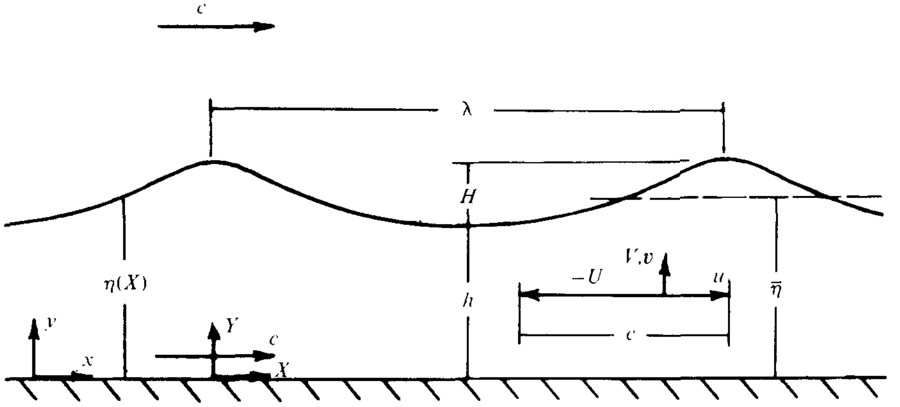


FIGURE 1. Typical steady water wave moving from left to right, showing the stationary co-ordinate system (x, y) and typical fluid velocities in this frame (u, v); the moving co-ordinate frame (X, Y) with typical velocities $(-U, V)$; and physical dimensions of the wave.

2. Exact operator equations

Consider two-dimensional periodic waves propagating from left to right without change of form over a layer of fluid on a horizontal bed. In a co-ordinate system (X, Y) with origin on the bed underneath a wave crest and moving with the same velocity as the waves, all motion is steady, with velocity components $(U(X, Y),\ V(X, Y))$ respectively.

Another co-ordinate system (x, y) is fixed on the bed, so that the waves move through it in the positive x direction, and the fluid velocity at any point is $(u(x,y,t), v(x,y,t))$. If $x_c(t)$ is the horizontal co-ordinate of the (X, Y) origin at any time t, and if the wave speed is $c = dx_c/dt$, then

$$x = x_c(t)+X, \quad y = Y, \quad u = U+c \quad \text{and} \quad v = V.$$

These co-ordinates and velocities are shown on figure 1, on which the important length scales are also shown; these are: λ = wavelength; H = wave height; h = minimum fluid depth, at the wave trough; $\eta(X)$ = fluid depth at any point; and $\bar{\eta}$ = mean fluid depth.

Throughout the following analysis we will deal only with steady motions in the (X, Y) frame, until we obtain expressions for velocities in both frames.

If the fluid motion is incompressible, a stream function $\psi(X, Y)$ exists such that the velocity components (U, V) are given by

$$U = \partial\psi/\partial Y \quad \text{and} \quad V = -\partial\psi/\partial X,$$

and, if the motion is irrotational, ψ satisfies Laplace's equation throughout the fluid:

$$\partial^2\psi/\partial X^2+\partial^2\psi/\partial Y^2 = 0. \tag{2.1}$$

The boundary conditions to be satisfied are:

(a) On the free surface, given by $Y = \eta(X)$, ψ is a constant, $-Q$ say.

(b) On the horizontal bed $Y = 0$, $\psi = 0$. Q is the total volume rate of flow per unit span in the steady flow. With the sign conventions we have chosen the apparent flow under the steady wave profile is from right to left.

(c) On the free surface, the pressure is constant, = 0 say, with Bernoulli's theorem giving

$$\tfrac{1}{2}((\partial\psi/\partial X)^2+(\partial\psi/\partial Y)^2)_{Y=\eta}+g\eta = R,$$

where R is the energy per unit mass of fluid in the steady flow and which is constant.

If ψ is assumed to be given by

$$\psi = -\sin YD.f(X) \tag{2.2}$$

where $\sin YD = \sin Yd/dX$ is the differential operator obtained by expanding in a power series:

$$\sin YD = Y\frac{d}{dX}-\frac{Y^3}{3!}\frac{d^3}{dX^3}+\frac{Y^5}{5!}\frac{d^5}{dX^5}-\dots,$$

and where $f'(X)$ is the horizontal fluid velocity on the bed, it is easily shown that

$$\partial\psi/\partial X = -\sin YD.f'(X), \quad \partial\psi/\partial Y = -\cos YD.f'(X),$$

$$\partial^2\psi/\partial X^2 = -\sin YD.f''(X), \quad \partial^2\psi/\partial Y^2 = \sin YD.f''(X),$$

and the field equation (2.1) is satisfied identically by (2.2). Also, substituting $Y = 0$ into (2.2) gives $\psi = 0$, satisfying the bottom boundary condition (b).

The kinematic surface boundary condition (a) is satisfied by substituting $Y = \eta(X)$, $\psi = -Q$ in (2.2):

$$Q = \sin\eta D.f(X). \tag{2.3}$$

Similarly the dynamic surface boundary condition (c) gives

$$\tfrac{1}{2}[(\sin\eta D.f')^2+(\cos\eta D.f')^2]+g\eta = R. \tag{2.4}$$

Equations (2.3) and (2.4) are two nonlinear, coupled ordinary differential equations in the unknowns $\eta(X)$, the fluid depth, and $f'(X)$, the horizontal velocity on the bed. In an earlier work Fenton (1972) inverted (2.3) to give an expression for f which was substituted into (2.4) to give an operator equation in terms of η and all its derivatives. However the expression involved a doubly infinite series: in the present work both (2.3) and (2.4) will be used to solve for $\eta(X)$ and $f'(X)$ together.

Now, η and Y are non-dimensionalized with respect to h, the minimum (or trough) depth of fluid. Other dimensions such as the mean depth $\bar{\eta}$ or wavelength λ could have been used; however, these give much longer expressions for all quantities (see §4.10). f is also non-dimensionalized with respect to Q such that (2.3) and (2.4) become

$$\sin\eta_* D_* . f_* = 1, \tag{2.5}$$

$$\tfrac{1}{2}[(\sin\eta_* D_* . f_*'(X_*))^2+(\cos\eta_* D_* . f_*'(X_*))^2]+g_*\eta_* = r_*, \tag{2.6}$$

where $X_* = X/h$, $\eta_* = \eta/h$, $D_* = d/dX_*$, $f_* = f/Q$, g_* is the 'gravity number' gh^3/Q^2, and r_* is the dimensionless energy, Rh^2/Q^2.

One of the infinite series of derivatives may be eliminated by differentiating (2.5):

$$D_*(\sin\eta_* D_* . f_*) = 0 = \sin\eta_* D_* . f_*' + \eta_*'\cos\eta_* D_* . f_*'.$$

This is the alternative form of the kinematic boundary condition, $V = Ud\eta/dX$, on the free surface. Substituting into (2.6) we have the following equations involving η_*, η_*' and odd derivatives only of f_*, and the two parameters g_* and r_*:

$$\sin\eta_* D_* . f_* - 1 = 0 \tag{2.7}$$

and

$$\tfrac{1}{2}(1+\eta_*'^2)(\cos\eta_* D_* . f_*')^2+g_*\eta_*-r_* = 0. \tag{2.8}$$

3. Series expansion solution

Equations (2.7) and (2.8) have the trivial solution of uniform flow with constant depth: $\eta_* = 1 = f'_* = g_*$, $r_* = \frac{3}{2}$, which is the well-known critical flow of hydraulic engineering. All quantities in (2.7) and (2.8) will be expanded about this state. The choice of expansion parameter is not obvious but, as the equations are nonlinear, we should consider all variation with X_* to be as αX_*, where α is a straining parameter (Lighthill 1949). Now, because even derivatives of f'_* occur in (2.7) and (2.8), terms like

$$D_*^{2n}(f'_*(\alpha X_*)) = (\alpha^2)^n f^{(2n+1)}(\alpha X_*)$$

occur, and there will be powers of α^2 throughout the system of equations. Hence it seems simplest to expand in terms of α^2 itself, and we write

$$\left.\begin{aligned} \eta_* &= 1+\sum_{i=1}^{\infty}(\alpha^2)^i Y_i(\alpha X_*),\\ f'_* &= 1+\sum_{i=1}^{\infty}(\alpha^2)^i F_i(\alpha X_*),\\ g_* &= 1+\sum_{i=1}^{\infty}(\alpha^2)^i g_i,\\ r_* &= \tfrac{3}{2}+\sum_{i=1}^{\infty}(\alpha^2)^i r_i, \end{aligned}\right\} \tag{3.1}$$

and substitute these into (2.7) and (2.8). Grouping all the terms in $\alpha^0, \alpha^2, \alpha^4, \ldots$, and requiring that the coefficient equation of each α^{2i} be satisfied identically, the following equations are obtained.

α^0: (2.7) and (2.8) satisfied identically.

$$\alpha^2\text{:}\quad F_1+Y_1 = 0; \tag{3.2a}$$

$$F_1+Y_1+g_1-r_1 = 0. \tag{3.2b}$$

These equations cannot be solved, but we do obtain

$$F_1 = -Y_1, \quad g_1 = r_1.$$

$$\alpha^4\text{:}\quad F_2+Y_2+F_1Y_1-\tfrac{1}{6}F''_1 = 0; \tag{3.2c}$$

$$F_2+Y_2+g_2-r_2-\tfrac{1}{2}F''_1+\tfrac{1}{2}F_1^2+g_1Y_1 = 0. \tag{3.2d}$$

Second-order terms appear at this stage – by subtracting one from the other they can be eliminated, and using (3.2*a, b*):

$$g_2-r_2-\tfrac{1}{3}F''_1+\tfrac{3}{2}F_1^2-g_1F_1 = 0. \tag{3.2e}$$

After some manipulation it can be shown that this has the solution

$$\left.\begin{aligned} F_1 &= -\tfrac{4}{3}m\,\mathrm{cn}^2(\alpha X_*|m),\\ g_1 &= \tfrac{4}{3}(1-2m) = r_1,\\ \text{and}\qquad r_2-g_2 &= \tfrac{8}{9}m(1-m), \end{aligned}\right\} \tag{3.2f}$$

where $\mathrm{cn}\,(\alpha X_*|m)$ is a Jacobian elliptic function of argument αX_* and parameter m. Often this is written $\mathrm{cn}\,(\alpha X_*, k)$ where $m = k^2$. Throughout the present work, no odd power of k is produced, hence we use m. For a description of these elliptic functions, see Abramowitz & Stegun (1964). The function $\mathrm{cn}\,(\alpha X_*|m)$ has a real period of $4K(m)$ where K is the complete elliptic integral of the first kind. Accordingly, $\mathrm{cn}^2\,(\alpha X_*|m)$ has a period of $2K(m)$, where m is obtained as a function of α by a solution of the equation

$$\alpha X_* = K(m) \quad \text{when} \quad X_* = \tfrac{1}{2}\lambda/h,$$

giving

$$\tfrac{1}{2}\alpha\lambda/h = K(m).$$

Substituting the solution (3.2f) into (3.1) and (3.2a) gives

$$\eta_* = 1+\tfrac{4}{3}m\alpha^2\,\mathrm{cn}^2\,(\alpha X_*|m)+O(\alpha^4).$$

Using the boundary condition at the crest: $\eta_*(0) = 1+H/h$, we obtain an expression for α in terms of H/h:

$$\alpha = \left(\frac{3}{4m}\frac{H}{h}\right)^{\frac{1}{2}}+O((H/h)^{\frac{3}{2}})$$

and

$$\eta_* = 1+(H/h)\,\mathrm{cn}^2(\alpha X_*|m)+O((H/h)^2),$$

the well-known solution of Korteweg & de Vries (1895), giving rise to the name 'cnoidal' waves.

Now considering α^6 terms of (2.7) and (2.8) into which (3.1) has been substituted we find the following:

$$\alpha^6\text{:}\; F_3+Y_3+F_2Y_1+Y_2F_1-\tfrac{1}{6}F_2''+A_3 = 0, \tag{3.2g}$$

$$F_3+Y_3+g_3-r_3+F_1F_2+Y_2g_1+Y_1g_2-\tfrac{1}{2}F_2''+B_3 = 0, \tag{3.2h}$$

where A_3 is a term involving first-order known quantities which give contributions at this $(\alpha^2)^3$ order, from (2.7):

$$A_3 = \tfrac{1}{120}F_1^{\mathrm{iv}}-\tfrac{1}{6}Y_1F_1'',$$

and B_3 is a similar term from (2.8):

$$B_3 = \tfrac{1}{24}F_1^{\mathrm{iv}}-Y_1F_1''-\tfrac{1}{2}F_1F_1''+\tfrac{1}{2}F_1'^2.$$

Subtracting (3.2h) from (3.2g),

$$\tfrac{1}{3}F_2''+Y_2(F_1-g_1)+F_2(Y_1-F_1)-g_2Y_1+r_3-g_3+A_3-B_3 = 0, \tag{3.3a}$$

a linear differential equation in the unknowns F_2, Y_2, r_3-g_3, in which (3.2c) can be used to eliminate Y_2. By assuming $F_2 = F_{20}+F_{21}\,\mathrm{cn}^2\,(\alpha X_*|m)+F_{22}\,\mathrm{cn}^4\,(\alpha X_*|m)$, substituting into (3.3$a$), and requiring that each power of cn^2 satisfy the equation, we obtain

$$F_2 = \tfrac{4}{9}m(m-1)+\tfrac{16}{9}m(1-2m)\,\mathrm{cn}^2+\tfrac{16}{9}m^2\,\mathrm{cn}^4,$$

$$Y_2 = \tfrac{8}{9}m(2m-1)\,\mathrm{cn}^2+\tfrac{4}{3}m^2\,\mathrm{cn}^4,$$

and

$$g_2 = -\tfrac{16}{45}+\tfrac{4}{5}m-\tfrac{4}{5}m^2,$$

where cn^2 is used to represent $\mathrm{cn}^2\,(\alpha X_*|m)$. Using the boundary condition that

$$\eta_*(0) = 1+H/h = 1+\alpha^2Y_1(0)+\alpha^4Y_2(0)$$

we recover the second-order solution of Laitone (1960):

$$\eta_* = 1+(H/h)\,\mathrm{cn}^2+(H/h)^2(-\tfrac{3}{4}\,\mathrm{cn}^2+\tfrac{3}{4}\,\mathrm{cn}^4)+O((H/h)^3), \tag{3.3b}$$

$$\alpha = \left(\frac{3}{4m}\frac{H}{h}\right)^{\frac{1}{2}}\left(1+\left(\frac{H}{h}\right)\frac{2-7m}{8m}\right)+O((H/h)^{\frac{5}{2}}). \tag{3.3c}$$

This method may be generalized and used at any order in the equations (2.7) and (2.8). If we have a solution correct to order $n-1$ so that we can write

$$\begin{Bmatrix} \eta_* \\ f_* \\ g_* \\ r_* \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ \frac{3}{2} \end{Bmatrix} + \sum_{i=1}^{n-1} (\alpha^2)^i \begin{Bmatrix} Y_i(\alpha X_*) \\ F_i(\alpha X_*) \\ g_i \\ r_i \end{Bmatrix}, \tag{3.4}$$

where all quantities on the right side are known, and this is substituted into (2.7) and (2.8), the following are obtained, similar to those for second order:

$$(\alpha^2)^{n+1}: F_{n+1} + Y_{n+1} - \tfrac{1}{6}F''_n + Y_n F_1 + F_n Y_1 + A_{n+1} = 0, \tag{3.5}$$

where the last two terms do not appear if $n = 2$. The quantity A_{n+1} is the coefficient of $(\alpha^2)^{n+1}$ in (2.7) obtained by substituting (3.4) correct to only $(\alpha^2)^{n-1}$ but formally manipulating to order $(\alpha^2)^{n+1}$. If B_{n+1} is the quantity obtained similarly from (2.8), we have

$$F_{n+1} + Y_{n+1} - \tfrac{1}{2}F''_n + Y_n g_1 + Y_1 g_n + F_1 F_n + g_{n+1} - r_{n+1} + B_{n+1} = 0,$$

and subtracting this from (3.5):

$$\tfrac{1}{3}F''_n + Y_n(F_1 - g_1) + F_n(Y_1 - F_1) - Y_1 g_n + r_{n+1} - g_{n+1} + A_{n+1} - B_{n+1} = 0, \quad n = 2, 3, \ldots. \tag{3.6}$$

To eliminate Y_n from this we use (3.5) obtained at one lower order:

$$F_n + Y_n - \tfrac{1}{6}F''_{n-1} + Y_{n-1}F_1 + F_{n-1}Y_1 + A_n = 0, \quad n = 2, 3, \ldots. \tag{3.7}$$

By assuming

$$F_n = \sum_{k=0}^{n} \sum_{l=0}^{n} F_{nkl} (\mathrm{cn}^2)^k m^l$$

the coefficients F_{nkl} may be found, and the whole solution at order n obtained $-F_n$, Y_n, g_n, r_n. The nth order solution is added to the expansions (3.4), and the whole procedure repeated at one higher order and so on.

The amount of algebraic manipulation is formidable, even at low order, and so computer programs were written to perform all the differentiation, addition, and multiplication of the triple series in α^2, cn^2 and m for η_* and f_* and the double series in α^2 and m for g_* and r_*. Results were obtained to ninth order in α^2, this being a reasonable limit for operation on the Cyber 72 computer at the University of N.S.W. All calculations were performed twice. Double-precision arithmetic correct to 28 significant figures was used in calculating coefficients of all the series to fifth order. The results were checked using an equation developed by Longuet-Higgins (see §4.6). The accuracy was such that single precision (14 figures) was adequate, and this was used in calculating results to ninth order.

4. Ninth-order cnoidal wave solution

4.1. *Choice of ϵ/m as expansion parameter*

While the expansions in α^2 were convenient for the work described in the previous section, it is most common in water wave theory for expansions to be presented in terms of dimensionless wave height: H/λ for deep water, and H/h for shallow water, the present case. To convert the series in α^2 to series with $\epsilon = H/h$ as the expansion quantity, the condition at the crest was used:

$$\eta_*(0)-1 = \frac{H}{h} = \epsilon = \sum_{i=1}^{9} (\alpha^2)^i Y_i(0).$$

This series was reverted by computer to give α^2 as a series in ϵ which was then substituted into each of the series in α^2 to give ninth-order expansions in ϵ for $\eta_*(X_*)$, $f'(X_*)$, g_* and r_*. Also, the square root of the series for α^2 was obtained, using the binomial theorem, to give the following result for α in which results to third order only are given:

$$\alpha = \left(\frac{3}{4}\frac{\epsilon}{m}\right)^{\frac{1}{2}}\left(1+\epsilon\left(-0{\cdot}875+\frac{0{\cdot}25}{m}\right)+\epsilon^2\left(0{\cdot}86719-\frac{0{\cdot}34375}{m}+\frac{0{\cdot}03125}{m^2}\right)+O(\epsilon^3)\right). \tag{4.1}$$

The parameter of the elliptic functions m can vary between 0 and 1: because m appears to negative powers throughout (4.1), if it does become small then some of the terms in (4.1) will be very large, and we should not expect the series to be useful. This variation of m can be studied using the expression (from § 3),

$$\alpha\frac{\lambda}{h} = 2K(m),$$

or substituting (4.1) to first order,

$$\left(\frac{3}{4}\frac{\epsilon}{m}\right)^{\frac{1}{2}}\frac{\lambda}{h} = 2K(m)+O(\epsilon^{\frac{3}{2}}).$$

This equation, although accurate only to first order, can be used to examine three limits of interest, giving some insight into the nature of the cnoidal solution.

(*a*) *Very long waves*. For finite wave height ϵ, as $\lambda/h \to \infty$ and the solitary wave case is approached, $K(m)$ must also approach ∞, requiring that $m \to 1$.

(*b*) *Short waves*. The elliptic integral $K(m)$ on the right side can never be zero, hence in the case of $\lambda/h \to 0$ the equation can only be satisfied by $m \to 0$, in which case $K \to \frac{1}{2}\pi$ and

$$\frac{m}{\epsilon} \to \frac{3}{4\pi^2}\frac{\lambda^2}{h^2}.$$

For finite wave height ϵ this shows explicitly how $m \to 0$ as $\lambda/h \to 0$, to first order at least.

(*c*) *Infinitesimal waves*. For a given wavelength λ/h using similar reasoning, as $\epsilon \to 0$, then it can be shown that m *also* goes to zero in the same way as in (*b*):

$$\epsilon/m \to \tfrac{1}{3}(2\pi h/\lambda)^2,$$

which is, in general, finite. Thus we have the result that, for very small waves, the parameter m also becomes small in such a way that ϵ/m remains finite.

If $\epsilon \to 0$ and $m \to 0$, then it is not clear which terms in (4.1) are important because ϵ appears in the numerator, and m in the denominator to different powers. This is immediately clarified if ϵ everywhere is associated with m^{-1}, so that (4.1) can be written in terms of (ϵ/m) as

$$\alpha = \left(\frac{3}{4}\frac{\epsilon}{m}\right)^{\frac{1}{2}}\left(1+\left(\frac{\epsilon}{m}\right)(0{\cdot}25-0{\cdot}875m)+\left(\frac{\epsilon}{m}\right)^2(0{\cdot}03125-0{\cdot}34375m+0{\cdot}867188m^2) + O((\epsilon/m)^3)\right). \tag{4.2}$$

From this it seems that a more natural expansion parameter is ϵ/m, even though the nominal one is ϵ. This equation contains positive powers only of ϵ/m and of m. The parameter m is now free to take on any of its possible values between 0 and 1, provided ϵ/m remains finite. For very long waves $m \to 1$ and the expansion parameter becomes ϵ. For infinitesimal waves, $\epsilon \to 0$, but as shown above ϵ/m is finite. This feature of the cnoidal expansion, that it is valid for infinitesimal and finite waves but that the effective expansion quantity ϵ/m is finite in both cases, is unusual. It seems that this is because it is based on a method by which even at lowest order a finite nonlinear solution is obtained: the series do not give more accurate results for infinitesimal waves. In Stokes wave expansions, however, the first-order solution is the linear one: the expansion quantity becomes small for infinitesimal waves and higher-order terms may be more easily neglected.

Of the three limiting cases (*a*), (*b*) and (*c*) above, only the short wave limit remains to be reconsidered in the light of the use of ϵ/m as the expansion parameter. For finite wave height ϵ, as $\lambda/h \to 0$, we showed above that $m/\epsilon \to 0$. Therefore, the use of ϵ/m makes it clear that for it to be small, and presumably for the series expansion to give accurate results, the short wave limit cannot be included. Throughout the rest of this paper, ϵ/m $(= H/mh)$ or $\bar{\epsilon}/m$ $(= H/m\bar{\eta})$ will be used as the expansion quantity in all series, explicitly showing the limitation of the present approach to waves which are neither too high nor too short.

The right side of (4.2) can be written in the form

$$\sum_{i=1}^{9} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j a_{ij},$$

where the a_{ij} are numerical coefficients. Similarly, all other series produced in the present work are of this form. In the equations containing variation with X_*, for example $\eta_*(X_*)$, each a_{ij} is a series in powers of $\mathrm{cn}^2(\alpha X_*|m)$. When expressions for fluid velocity at a point are given each a_{ij} is a double series in powers of cn^2 and of Y_*^2. As will be seen, when integrated quantities such as $\bar{\eta}_*$ are obtained, the a_{ij} are a single series in powers of elliptic integrals.

4.2. *Use of minimum depth h as the depth scale*

In Stokes wave expansions the mean depth $\bar{\eta}$ is used to represent water depth, while minimum or trough depth has been used for cnoidal waves. Mean depth seems to be a more fundamental scale but it will now be shown that using $\bar{\eta}$ in the present work would introduce extra series in powers of elliptic integrals requiring the specification of many more coefficients in the presentation of a solution.

From § 3, and as described in § 4.1, a series for η/h was obtained:

$$\frac{\eta}{h} = \eta_* = 1 + \sum_{i=1}^{9} \left(\frac{\epsilon}{m}\right)^i \sum_{j=1}^{i} m^j \sum_{k=1}^{j} \mathrm{cn}^2(\alpha X_*|m)\, \eta_{ijk},$$

where the η_{ijk} are numerical coefficients. The dimensionless mean depth $\bar{\eta}_* = \bar{\eta}/h$ was obtained by integrating this expression:

$$\bar{\eta} = \frac{1}{\lambda}\int_0^{\lambda} \eta(X)\,dX,$$

or, in dimensionless terms,

$$\bar{\eta}_* = \frac{\bar{\eta}}{h} = \frac{1}{K}\int_0^K \eta_*(\alpha X_*)\, d(\alpha X_*),$$

where K is the complete elliptic integral of the first kind, $K(m)$. A computer program was written to integrate series in cn^2, using a recursion relation from Gradshteyn & Ryzhik (1965, §5.13):

$$\text{if} \quad I(l) = \frac{1}{K}\int_0^K [m\,\mathrm{cn}^2(\theta|m)]^l\, d\theta,$$

then

$$I(0) = 1 \quad \text{and} \quad I(1) = -1 + m + (E/K),$$

where E is the complete elliptic integral of the second kind, $E(m)$, and

$$I(l+2) = \left(\frac{2l+2}{2l+3}\right)(2m-1)\,I(l+1) + \left(\frac{2l+1}{2l+3}\right)(m-m^2)\,I(l). \tag{4.3}$$

In this way we see that $\bar{\eta}_*$ is a double series in ϵ/m and m, but with each term containing one numerical coefficient plus a second coefficient of E/K:

$$\bar{\eta}_* = \frac{\bar{\eta}}{h} = 1 + \sum_{i=1}^{9}\left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j(\bar{\eta}_{ij1} + \bar{\eta}_{ij2}\,E(m)/K(m)). \tag{4.4}$$

While this doubles the number of coefficients necessary, the situation is worse if, for example, $\bar{\eta}$ is to be used in the denominator as a depth scale. Such a case arises if we want to give an expression for the dimensionless wave height $\bar{\epsilon} = H/\bar{\eta}$, as used in Stokes type expansions:

$$\bar{\epsilon} = \frac{H}{\bar{\eta}} = \frac{H}{h}\Big/\frac{\bar{\eta}}{h} = \epsilon\bar{\eta}_*^{-1}. \tag{4.5}$$

Using the binomial expansion to evaluate $\bar{\eta}_*^{-1}$ from (4.4) it can be seen that every coefficient of $(\epsilon/m)^i m^j$ is another series in $(E/K)^k$, $k = 0, \ldots, i$, thereby greatly increasing the number of coefficients necessary in a solution. For this reason, the trough or minimum depth h is used in this work as much as possible in the presentation of results. However, it is the mean water depth which is known when a real problem is specified. In the next section, expressions are given so that h may be obtained from $\bar{\eta}$ and subsequently used as the depth scale.

4.3. *Obtaining a cnoidal wave solution given water depth, wave height, and wave length or period*

The shallow water approximations given in this work generally use h as the depth scale, so that the series can be evaluated if both $\epsilon = H/h$ and the parameter m are known. Neither are known initially, however, as commonly a wave height H, mean water depth $\bar{\eta}$, and either wavelength λ or period τ are specified.

Using computer manipulation of the series the expansion (4.5) was obtained:

$$\bar{\epsilon} = \frac{H}{\bar{\eta}} = \sum_{i=1}^{9}\left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i}\left(\frac{E}{K}\right)^k e_{ijk},$$

where the e_{ijk} are numerical coefficients. This quantity $\bar{\epsilon}$ could have been used throughout, which would have required the presentation of many more coefficients. Instead, as it is known, *a priori*, it is used only in this section to give equations for ϵ and m. This equation was then reverted to give ϵ as a series in $\bar{\epsilon}$ and m:

$$\epsilon = \sum_{i=1}^{9} \left(\frac{\bar{\epsilon}}{m}\right) \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \left(\frac{E}{K}\right)^k \bar{e}_{ijk},$$

where the $\bar{e}_{ijk}$ are coefficients. This was substituted into the full ninth-order version of (4.2) to give α as a function of $\bar{\epsilon}$, m, and $E(m)/K(m)$, and into (4.4) to give $\bar{\eta}_*$ as a function of the same quantities. From § 4.1 the equation for wavelength was used:

$$\alpha\lambda/h = 2K(m),$$

therefore
$$\bar{\lambda}_* = \frac{\lambda}{\bar{\eta}} = \frac{2K(m)}{\alpha\bar{\eta}_*},$$

and the expansions for α and $\bar{\eta}_*$ substituted, the subsequent expansion inverted to give:

$$\bar{\lambda}_* = \frac{\lambda}{\bar{\eta}} = \frac{4K(m)}{(3\bar{\epsilon}/m)^{\frac{1}{2}}} \sum_{i=0}^{8} \left(\frac{\bar{\epsilon}}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \left(\frac{E(m)}{K(m)}\right)^k \lambda_{ijk}, \tag{4.6}$$

where the λ_{ijk} are coefficients, as given to fifth order in the appendix, table A 1.

The significance of this equation is that it enables us to solve for m, provided the three length quantities are known: wavelength λ, mean depth $\bar{\eta}$ and wave height H, giving $\bar{\lambda}_* = \lambda/\bar{\eta}$ and $\bar{\epsilon} = H/\bar{\eta}$. Clearly (4.6) is an implicit equation for m, in which it is too deeply embedded for direct solution. It could be solved by trial and error, or by Newton's method. The author used trial and error in obtaining the results of § 5.

In many practical problems it is not the wavelength that is known initially, but rather the wave period τ. In this case an equation similar to (4.6) can be obtained from the relation $\tau = \lambda/c$, where c is the phase speed of the waves. A dimensionless period τ_* can be introduced as $\tau_* = \tau(g/\bar{\eta})^{\frac{1}{2}}$, giving

$$\tau_* = \tau(g/\bar{\eta})^{\frac{1}{2}} = \bar{\lambda}_*(g\bar{\eta})^{\frac{1}{2}}/c.$$

The dimensionless wave speed $c/(g\bar{\eta})^{\frac{1}{2}}$ was obtained, using the methods described in the next section, as an expansion in $\bar{\epsilon}$, m, and E/K, and substituted into the above equation to give

$$\tau_* = \frac{4K(m)}{(3\bar{\epsilon}/m)^{\frac{1}{2}}} \sum_{i=0}^{8} \left(\frac{\bar{\epsilon}}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \left(\frac{E}{K}\right)^k \tau_{ijk}, \tag{4.7}$$

in which the τ_{ijk} are numerical coefficients given in table A 2 in the appendix to fifth order. If water depth $\bar{\eta}$, wave height H and wave period τ are known, then this implicit equation for m may be solved. A similar procedure must be followed in high-order Stokes wave problems, when an implicit equation has to be solved for the quantity $\bar{\eta}/\lambda$, when λ is not known *a priori*.

If m has been calculated for a particular problem, h remains to be found. This can quickly be done using the equations obtained during the development of (4.6), which gave the equation

$$\frac{h}{\bar{\eta}} = 1 + \sum_{i=1}^{9} \left(\frac{\bar{\epsilon}}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \left(\frac{E}{K}\right)^k h_{ijk}, \tag{4.8}$$

where the h_{ijk} are numerical coefficients given in table A 3. From the value of $h/\bar{\eta}$, ϵ is given:

$$\epsilon = \bar{\epsilon}\bigg/\left(\frac{h}{\bar{\eta}}\right),$$

and the two quantities (ϵ/m) and m so found may be used in all subsequent expressions.

4.4. *Presentation of expansions obtained*

All series for the physical quantities given below were obtained to ninth order by computer manipulation, however the number of coefficients necessary to specify these expansions completely is so large that it is not feasible here. Instead, all coefficients are given to fifth order in (ϵ/m) as set out in the appendix. Subsequent comparison with experimental results (see § 5) showed that including higher-order terms gave no better agreement. The coefficients are given as real numbers, rounded to five decimal places. This is sufficiently accurate for direct evaluation of the series; however, further manipulation of the series accurate to this degree could produce unacceptable round-off errors.

Throughout the following description of these results, numerical coefficients in a particular expansion for a physical quantity will be represented by the symbol for that quantity with the number of subscripts necessary, for example α_{ij} in the expansion for α.

Coefficient of X_*: α.

$$\alpha = \left(\frac{3}{4}\frac{\epsilon}{m}\right)^{\frac{1}{2}} \sum_{i=0}^{4} \left(\frac{\epsilon}{m}\right)^{i} \sum_{j=0}^{i} m^j \alpha_{ij}. \quad \text{(See appendix table B 1 for the } \alpha_{ij}\text{)}$$

For the solitary wave case, $m = 1$, the results of Fenton (1972) are obtained. Throughout the present work the $m = 1$ case reduced correctly to the solitary wave results given by Fenton.

Wave profile: η.

$$\frac{\eta}{h} = \eta_* = 1 + \sum_{i=1}^{5} \left(\frac{\epsilon}{m}\right)^{i} \sum_{j=1}^{i} m^j \sum_{k=1}^{j} \mathrm{cn}^2\,(\alpha X_*|m)\,\eta_{ijk}. \qquad \text{(Table B 2)}$$

Volume flux: Q. The flux, per unit span normal to the plane of flow, under the wave and relative to the wave was obtained from the results for g_*:

$$Q_* = \frac{Q}{(gh^3)^{\frac{1}{2}}} = g_*^{-\frac{1}{2}} = 1 + \sum_{i=1}^{5} \left(\frac{\epsilon}{m}\right)^{i} \sum_{j=0}^{i} m^j Q_{ij}. \qquad \text{(Table B 3)}$$

Energy per unit mass: R. This is the Bernoulli constant in the steady flow as given in (2.4). An expansion for $r_* = Rh^2/Q^2$ was obtained as described in § 3. This was multiplied by Q_*^2 to give the dimensionless R_*:

$$R_* = \frac{R}{gh} = r_* Q_*^2 = 1{\cdot}5 + \sum_{i=1}^{5} \left(\frac{\epsilon}{m}\right)^{i} \sum_{j=0}^{i} m^j R_{ij}. \qquad \text{(Table B 4)}$$

Fluid velocity relative to wave: $U(X, Y)$, $V(X, Y)$. From the differential operator of § 2 we have

$$U = \partial\psi/\partial y = -\cos YD.f'(X),$$

or, in dimensionless terms,

$$Uh/Q = -\cos Y_* D_* . f'_*(X_*),$$

and multiplying by Q_*:

$$U/(gh)^{\frac{1}{2}} = -Q_* \cos Y_* D_* . f'_*(X_*).$$

Substituting the results of § 3 for f'_* we obtain a quadruple series in (ϵ/m), (Y/h), m, and $\mathrm{cn}^2(\alpha X_*|m)$, for $U(X,Y)/(gh)^{\frac{1}{2}}$, the horizontal fluid velocity relative to the wave. With the conventions that we have chosen, waves progressing to the right over otherwise quiescent fluid, in the moving frame the steady fluid velocity will be from right to left, under the 'stationary' wave profile, and will be negative. In practical applications it is not this velocity which is important, but rather the velocity relative to a stationary frame, $u(x, y, t)$. After the wave speed is calculated, we will use the results of the present section to give expressions for unsteady velocities, accelerations and pressures throughout the fluid.

$$\frac{U}{(gh)^{\frac{1}{2}}} = -1 - \sum_{i=1}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \mathrm{cn}^{2k}(\alpha X_*|m) \sum_{l=0}^{i-1} \left(\frac{Y}{\eta}\right)^{2l} \phi_{ijkl}.$$

At fifth order this series has 200 terms, given in table B 5, and at ninth 1449 terms! Evaluation by hand calculation is not practicable, however on a computer it is trivial. To obtain the vertical component of fluid velocity we can use the equation that the fluid motion is incompressible,

$$\partial U/\partial X + \partial V/\partial Y = 0.$$

That is,

$$\frac{V}{(gh)^{\frac{1}{2}}} = \int_0^{Y_*} \frac{\partial}{\partial X_*}\left[\frac{-U}{(gh)^{\frac{1}{2}}}\right] dY_*,$$

and with $V(X_*, 0) = 0$:

$$\frac{V}{(gh)^{\frac{1}{2}}} = -2\alpha\, \mathrm{cn}\,(\alpha X_*|m)\, \mathrm{sn}\,(\alpha X_*|m)\, \mathrm{dn}\,(\alpha X_*|m) \sum_{i=1}^{5}\left(\frac{\epsilon}{m}\right)^i \sum_{i=1}^{i} m^j \sum_{k=1}^{i} (\mathrm{cn}^2(\alpha X_*|m))^{k-1} \times \sum_{l=0}^{i-1}\left(\frac{Y}{h}\right)^{2l+1}\left(\frac{k}{2l+1}\phi_{ijkl}\right).$$

The elliptic functions sn and dn are simply obtained from cn:

$$\mathrm{sn}^2(\theta|m) = 1 - \mathrm{cn}^2(\theta|m)$$

and

$$\mathrm{dn}^2(\theta|m) = 1 - m\,\mathrm{sn}^2(\theta|m),$$

which appeared here after differentiation of cn:

$$\frac{d}{d\theta}(\mathrm{cn}\,(\theta|m)) = -\mathrm{sn}\,(\theta|m)\,\mathrm{dn}\,(\theta|m).$$

The equation for $V/(gh)^{\frac{1}{2}}$ is not a formally correct expansion in (ϵ/m), as it contains the summations as shown, all of which are multiplied by α, which is itself an expansion in ϵ/m. To give the correct expansion obtained by multiplication would yield another large number of coefficients which would have to be presented separately. It does not seem reasonable to do this. Below it will be shown how fluid pressures and accelerations can be obtained from the same set of ϕ_{ijkl}.

Wave speed: c. This is important in relating velocities in the two co-ordinate frames. Wave speed is defined to be the time mean horizontal fluid velocity at a fixed point in the stationary frame, equal to the spatial mean at a level wholly within the fluid in the moving frame. That is,

$$c = \frac{1}{\lambda}\int_0^\lambda U(X, Y)\,dX,$$

in which the negative sign of U in the present convention is ignored. Level $Y = 0$ was used to evaluate this integral, using (4.3):

$$c_* = \frac{c}{(gh)^{\frac{1}{2}}} = \frac{1}{K}\int_0^K \frac{U}{(gh)^{\frac{1}{2}}}(\alpha X_*, 0)\,d(\alpha X_*) = 1 + \sum_{i=1}^{5}\left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{1}\left(\frac{E}{K}\right)^k c_{ijk}.$$

(Table B 6)

The elliptic integral ratio $E(m)/K(m)$ entered here because of the integration of powers of cn^2.

Fluid velocity in stationary frame: $u(x, y, t)$, $v(x, y, t)$. In most practical applications it is the velocity relative to a fixed point as waves pass which is more important than that relative to moving axes. If (x, y) are the co-ordinates of a point in the stationary frame at which the velocity components are (u, v) then

$$x = X + x_c(t), \quad x_* = x/h = X_* + x_c^*(t), \quad y = Y, \quad y_* = Y_*,$$

where $x_c(t)$ is the horizontal distance of the wave crest from the origin such that $c = dx_c/dt$,

$$u = U + c \quad \text{and} \quad v = V.$$

Substituting the results from U and V, we note that the vertical velocity is the same in both frames:

$$\frac{v}{(gh)^{\frac{1}{2}}}(x, y, t) = \frac{V}{(gh)^{\frac{1}{2}}}(X, Y),$$

where the argument of all elliptic functions is now $\alpha(x_* - x_c^*(t))$ instead of αX_*. The horizontal velocity u is

$$\frac{u(x_*, y_*, t)}{(gh)^{\frac{1}{2}}} = c_* - 1 - \sum_{i=1}^{5}\left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \mathrm{cn}^{2k}(\alpha(x_* - x_c^*(t))|m) \sum_{l=0}^{i-1}\left(\frac{y}{h}\right)^{2l} \phi_{ijkl}.$$

Fluid acceleration: $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial t$, Du/Dt, $\partial v/\partial x$, $\partial v/\partial y$, $\partial v/\partial t$, Dv/Dt. In many applications the temporal and spatial derivatives of velocity are required, such as in the use of empirical drag force laws. These are simply obtained from the above equations and from continuity and irrotationality conditions:

$$\left(\frac{h}{g}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x} = -\left(\frac{h}{g}\right)^{\frac{1}{2}}\frac{\partial v}{\partial y} = 2\alpha\,\mathrm{cn}.\mathrm{sn}.\mathrm{dn}\sum_{i=1}^{5}\left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=1}^{i} (\mathrm{cn}^2)^{k-1} \sum_{l=0}^{i-1}\left(\frac{y}{h}\right)^{2l}(k\phi_{ijkl}),$$

$$\left(\frac{h}{g}\right)^{\frac{1}{2}}\frac{\partial u}{\partial y} = \left(\frac{h}{g}\right)^{\frac{1}{2}}\frac{\partial v}{\partial x} = -2\sum_{i=1}^{5}\left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j \sum_{k=0}^{i} \mathrm{cn}^{2k} \sum_{l=1}^{i-1}\left(\frac{y}{h}\right)^{2l-1}(l\phi_{ijkl}),$$

in which all elliptic functions have argument $\alpha(x_* - x_c^*(t))$,

$$\frac{1}{g}\frac{\partial u}{\partial t} = -c_*\left(\frac{h}{g}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{1}{g}\frac{\partial v}{\partial t} = -c_*\left(\frac{h}{g}\right)^{\frac{1}{2}}\frac{\partial v}{\partial x}.$$

The acceleration of a fluid particle can be obtained simply from these expressions:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}.$$

Pressure in the fluid: $p(x,y,t)$. The fluid pressure p would require another set of coefficients as numerous as the ϕ_{ijkl} if it were to be provided formally. However, these ϕ_{ijkl} may be used to obtain an expression for the pressure. At any point in the fluid, Bernoulli's theorem holds in the frame in which motion is steady:

$$p(X,\ Y)/\rho+\tfrac{1}{2}(U^2+V^2)+gY = R.$$

Dividing by gh, and noting that $X = x-x_c(t)$, $Y = y$,

$$p(x,y,t)/\rho gh = R/gh-y_*-\tfrac{1}{2}[U^2/gh+V^2/gh],$$

where R/gh, $U/(gh)^{\frac{1}{2}}$ and $V/(gh)^{\frac{1}{2}}$ have all been given as fifth-order expansions above. The argument of all elliptic functions is $\alpha X_* = \alpha(x_*-x_c^*(t))$.

4.5. *Integral properties of cnoidal waves*

In most areas of engineering application, important quantities are those which, as functions of position and time, cause dynamic loads on structures. These include the surface elevation and the fluid pressures, velocities and accelerations, for which expansions have already been given. As well as these there are a number of integral properties of the periodic waves which may be of interest. In § 4.4 expressions have been given for some of these: the volume flux $Q/(gh^3)^{\frac{1}{2}}$, energy per unit mass R/gh, and $c/(gh)^{\frac{1}{2}}$ the wave speed. The work of Longuet-Higgins (1975) enables us to calculate several other integral properties rather more easily than would otherwise be possible, and frequent reference will be made to that work. Equation numbers with the prefix L-H refer to the equation numbers in Longuet-Higgins (1975). Instead of using $\rho = 1$, we will include it explicitly. All overbars in the following denote a mean with respect to X.

Computer programs were written to perform all necessary manipulations in this section. The following expansions in (ϵ/m) were obtained.

Wave impulse: I. The mean wave impulse per unit horizontal area is

$$I = \overline{\int_0^\eta \rho u\, dy} = \rho(c\bar{\eta}-Q), \qquad \text{(L-H: A)}$$

and

$$I_* = \frac{I}{\rho(gh^3)^{\frac{1}{2}}} = c_*\bar{\eta}_*-Q_* = \sum_{i=2}^{5}\left(\frac{\epsilon}{m}\right)^i\sum_{j=0}^{i-1}m^j\sum_{k=0}^{2}\left(\frac{E}{K}\right)^k I_{ijk}. \qquad \text{(Table C 1)}$$

In common with most other integral quantities given below, the first contribution to wave impulse is at second order in (ϵ/m): the first-order solution gives no contribution.

Kinetic energy: T. The mean kinetic energy per unit horizontal area is

$$T = \overline{\int_0^\eta \tfrac{1}{2}\rho(u^2+v^2)\,dy} = \tfrac{1}{2}cI, \qquad \text{(L-H: B, after Lévi-Civita)}$$

and

$$T_* = T/\rho gh^2 = \tfrac{1}{2}c_*I_* = \sum_{i=2}^{5}\left(\frac{\epsilon}{m}\right)^i\sum_{j=0}^{i-1}m^j\sum_{k=0}^{3}\left(\frac{E}{K}\right)^k T_{ijk}. \qquad \text{(Table C 2)}$$

Potential energy: V. The conventional symbol V is used, which has already represented the vertical velocity; the two quantities are so different that the ambiguity should not matter. Mean potential energy per unit horizontal area is

$$V = \overline{\int_{\bar{\eta}}^{\eta}\rho g(y-\bar{\eta})\,dy} = \tfrac{1}{2}\rho g(\overline{\eta^2}-\bar{\eta}^2),$$

therefore

$$V_* = V/\rho gh^2 = \tfrac{1}{2}(\overline{\eta_*^2}-\bar{\eta}_*^2).$$

To obtain the first term $\overline{\eta_*^2}$, the series for η_* was squared and then integrated in the same way that $\bar{\eta}_*$ was produced:

$$V_* = \sum_{i=2}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i-1} m^j \sum_{k=0}^{2} \left(\frac{E}{K}\right)^k V_{ijk}. \qquad \text{(Table C 3)}$$

Mean square of bed velocity: $\overline{u_b^2}$. This quantity may be used in the estimation of real fluid effects. After Longuet-Higgins (1975) we have

$$\overline{u_b^2} = \frac{1}{\lambda}\int_0^\lambda u^2(x,0,t)\,dx = 2(R-g\bar{\eta})-c^2, \qquad \text{(L-H: 3.2, 3.6)}$$

giving
$$\overline{u_b^2}/gh = 2(R/gh-\bar{\eta}_*)-c_*^2,$$

which was evaluated to yield

$$\frac{\overline{u_b^2}}{gh} = \sum_{i=2}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i-1} m^j \sum_{k=0}^{2} \left(\frac{E}{K}\right)^k U_{ijk}. \qquad \text{(Table C 4)}$$

Radiation stress: S_{xx}. The excess flux of momentum per unit span due to the waves is the radiation stress:

$$S_{xx} = \overline{\int_0^\eta (p+\rho u^2)\,dy} - \tfrac{1}{2}\rho g\bar{\eta}^2 = 4T-3V+\bar{\eta}\overline{u_b^2}. \qquad \text{(L-H: C)}$$

Therefore,
$$S_{xx}^* = \frac{S_{xx}}{\rho g h^2} = 4T_*-3V_*+\bar{\eta}_*\frac{\overline{u_b^2}}{gh},$$

which gave
$$S_{xx}^* = \sum_{i=2}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i-1} m^j \sum_{k=0}^{3} \left(\frac{E}{K}\right)^k (S_{xx})_{ijk}. \qquad \text{(Table C 5)}$$

Mean wave power: F. The mean energy flux or wave power per unit span is

$$F = \overline{\int_0^\eta [p+\tfrac{1}{2}\rho(u^2+v^2)+\rho g(y-\eta)]\,u\,dy} = (3T-2V)c+\tfrac{1}{2}\overline{u_b^2}(I+\rho c\bar{\eta}), \qquad \text{(L-H: 3.10)}$$

which becomes, in dimensionless terms,

$$F_* = \frac{F}{\rho(g^3h^5)^{\frac{1}{2}}} = (3T_*-2V_*)c_*+\frac{1}{2}\frac{\overline{u_b^2}}{gh}(I_*+c_*\bar{\eta}_*),$$

and upon substitution of the various series gave

$$F_* = \sum_{i=2}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i-1} m^j \sum_{k=0}^{4} \left(\frac{E}{K}\right)^k F_{ijk}. \qquad \text{(Table C 6)}$$

This series contained terms up to $(E/K)^4$, the highest power in all the physical quantities except for inverted series such as $h/\bar{\eta}$.

Mean Stokes drift velocity: C_s. Stokes' second definition of wave speed, the mean velocity throughout the fluid in the translating co-ordinate system, is

$$C = \frac{\int_0^\lambda\int_0^\eta U\,dY\,dX}{\int_0^\lambda\int_0^\eta dY\,dX} = \frac{Q}{\bar{\eta}}.$$

Therefore,
$$\frac{C}{(gh)^{\frac{1}{2}}} = \frac{Q_*}{\bar{\eta}_*},$$

and subtracting the wave speed c_* we obtain the mean speed at which the fluid particles move, relative to a fixed frame, which is the mean Stokes drift velocity, $C_s/(gh)^{\frac{1}{2}}$:

$$C_s^* = C_s/(gh)^{\frac{1}{2}} = Q_*/\overline{\eta}_* - c_*,$$

which was evaluated to give

$$C_s^* = \sum_{i=2}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i-1} m^j \sum_{k=0}^{1} \left(\frac{E}{K}\right)^k C_{ijk}. \qquad \text{(Table C 7)}$$

Momentum flux: S. The momentum flux per unit span in the steady flow, S, is one of the three parameters Q, R and S introduced by Benjamin & Lighthill (1954) in their study of cnoidal waves. Expressions for dimensionless Q and R were given above; here we use another result of Longuet-Higgins (1975) to obtain an expression for S:

$$S = \int_0^\eta (p + \rho U^2)\, dY = S_{xx} - 2cI + \rho\overline{\eta}(c^2 + \tfrac{1}{2}g\overline{\eta}), \qquad \text{(L-H: 5.2)}$$

which in dimensionless terms becomes

$$S_* = S_{xx}^* - 2c_* I_* + \overline{\eta}_*(c_*^2 + \tfrac{1}{2}\overline{\eta}_*).$$

This was evaluated and it was found that all the terms in E/K cancelled, leaving the relatively simple expression

$$S_* = \tfrac{3}{2} + \sum_{i=1}^{5} \left(\frac{\epsilon}{m}\right)^i \sum_{j=0}^{i} m^j S_{ij}. \qquad \text{(Table C 8)}$$

The disappearance of E/K from this series might have been expected because of the close relation between Q, R and S and the fact that neither Q nor R contained E/K. In this way it provides something of a check on all the series used in generating it.

4.6. *Check on results*

Several of the series presented can be checked using the following equation, developed by Longuet-Higgins using variational techniques:

$$d(T - V) = 2T\, dc/c + (T - 2V + \tfrac{1}{2}\overline{u_b^2}\eta)\, d\lambda/\lambda - \tfrac{1}{2}\overline{u_b^2}\, d\overline{\eta}, \qquad \text{(L-H: 4.17)}$$

where the differentials can be due to changes in wave height, length and mean depth. All quantities in this equation have been given previously as expansions in (ϵ/m), m, and E/K, in which m is deeply embedded. Thus, it seems reasonable to keep m constant and to vary ϵ only. Wavelength λ is given by the equation from § 3:

$$\alpha\lambda/h = 2K(m),$$

therefore, keeping m constant,

$$\frac{1}{\alpha}\frac{\partial \alpha}{\partial \epsilon} + \frac{1}{\lambda}\frac{\partial \lambda}{\partial \epsilon} = 0,$$

and substituting this and all our dimensionless quantities,

$$\left(\frac{\partial T_*}{\partial \epsilon} - \frac{\partial V_*}{\partial \epsilon} - I_* \frac{\partial c_*}{\partial \epsilon} + \frac{1}{2}\frac{\overline{u_b^2}}{gh}\frac{\partial \overline{\eta}_*}{\partial \epsilon}\right)\alpha + \left(T_* - 2V_* + \frac{1}{2}\frac{\overline{u_b^2}}{gh}\eta_*\right)\frac{\partial \alpha}{\partial \epsilon} = 0.$$

	Double precision accuracy	Single precision accuracy
(ϵ/m)	0	$0{\cdot}4 \times 10^{-14}$
$(\epsilon/m)^2$	$0{\cdot}4 \times 10^{-27}$	$0{\cdot}6 \times 10^{-13}$
$(\epsilon/m)^3$	$0{\cdot}3 \times 10^{-26}$	$0{\cdot}2 \times 10^{-12}$
$(\epsilon/m)^4$	$0{\cdot}2 \times 10^{-25}$	$0{\cdot}2 \times 10^{-11}$
$(\epsilon/m)^5$	$0{\cdot}8 \times 10^{-25}$	$0{\cdot}7 \times 10^{-11}$
$(\epsilon/m)^6$	—	$0{\cdot}5 \times 10^{-10}$
$(\epsilon/m)^7$	—	$0{\cdot}4 \times 10^{-9}$
$(\epsilon/m)^8$	—	$0{\cdot}2 \times 10^{-8}$
$(\epsilon/m)^9$	—	$0{\cdot}1 \times 10^{-7}$

TABLE 1

The series expansions generated by the computer programs were substituted into this equation and the maximum error at each order obtained, such that of all coefficients of m^j and $(E/K)^k$ at each order in (ϵ/m) the maximum error is shown in table 1.

Clearly, the coefficients used in the series are accurate and there has been no disastrous loss of significance. It does seem, however, that the ninth order is approaching the limit of single precision accuracy.

5. Comparison with previous work

5.1. *Stokes wave expansions*

All results in this work have been presented as double series in powers of (ϵ/m) and of m. For the series to be truncated at finite order and yet give accurate results, (ϵ/m) must be not large. As shown in §4.1, this is given by $\epsilon = H/h$ being small, but the parameter m being finite. If m does become small, however, it was shown that, as $m \to 0$,

$$\epsilon/m \to \tfrac{4}{3}\pi^2 h^2/\lambda^2,$$

showing how ϵ/m can become large for short waves.

Stokes wave expansions which explicitly contain the water depth (e.g. Skjelbreia & Hendrickson 1961) have similar features to the series presented here, but are, in a sense, complementary. The nominal Stokes expansion parameter is ak, where a is approximately half the wave height and k is the wavenumber $2\pi/\lambda$, so that the expansion parameter is effectively $\pi H/\lambda$. However, the denominator of each term contains powers of $\sinh k\bar{\eta}$, so that the ratio of successive terms in the expansion can be shown to be like

$$\frac{\pi H}{\lambda}\frac{1}{\sinh^3 k\bar{\eta}},$$

where $\bar{\eta}$ is the mean water depth. For waves in deep water no problems arise; however, for shallow water $k\bar{\eta} \to 0$, and the effective expansion parameter is

$$\frac{1}{8\pi^2}\frac{H}{\lambda}\left(\frac{\lambda}{\bar{\eta}}\right)^3, \quad \text{or} \quad \frac{1}{8\pi^2}\bar{\epsilon}\left(\frac{\lambda}{\bar{\eta}}\right)^2,$$

showing that for large values of $\lambda/\bar{\eta}$, the long wave case, this quantity can become large: precisely the reverse for the cnoidal wave expansions.

These deep water, or Stokes wave, expansions have often been applied in the past, in the absence of an accurate shallow water theory, to long waves for which they are not valid. This misapplication could have been avoided if Stokes expansions had been presented as a double series in $(ak)/\sinh^3 k\bar{\eta}$ and in $\sinh^2 k\bar{\eta}$, for example,

$$\sum_{i=1}^{5}\left(\frac{ak}{\sinh^3 k\bar{\eta}}\right)^i \sum_{j=0}^{i} (\sinh k\bar{\eta})^{2j}\, b_{ij}.$$

By using $(ak/\sinh^3 k\bar{\eta})$ one would be warned if the expansion were being applied to long waves in the same way that using ϵ/m in the cnoidal wave expansion would show if it were being applied to short waves.

5.2. *Validity of expansions for speed wave power, etc.*

The most accurate and comprehensive results for progressive gravity waves are those of Cokelet (1977) for waves up to and including the highest and covering most of the range between deep and shallow water. His results were obtained for ten values of dimensionless depth and for each depth some thirty wave amplitudes. These were obtained from a very high-order Stokes series (Schwartz 1974) for which accurate sums were found using Padé approximants. If accurate values of integral parameters such as wave speed, or potential energy are required, these could be interpolated from Cokelet's tables. Because of the very high order of the series and the numerical solution necessary, explicit expressions were not given for any quantities; however, the numerical results given provide an accurate basis for comparing the results of Stokes and cnoidal expansions. Cokelet's results (1977) are more accurate than this work could hope to match, so there seems little point in presenting a large number of tables and graphs of all the results: the results of the present work are the tables of coefficients of the series which may be applied to *any* shallow water situation.

It does not seem necessary to make detailed comparisons of all the wave quantities for which results have been given in appendix C. As can be seen in Cokelet's paper the gross behaviour of each as affected by wave height and length is similar. Instead, only one quantity will be compared here: the wave speed, which has traditionally been used as the first basis for comparison between wave theories. All relevant quantities were taken from Cokelet's tables and for a certain constant value of 'equivalent depth' d/λ, where $d = Q/c$, pairs of values of dimensionless wave height $H/\bar{\eta}$ and dimensionless speed $c^2/g\bar{\eta}$ recorded. Then, knowing $H/\bar{\eta}$ and $\lambda/\bar{\eta}$, m was found from table A 1 by trial and error, h found from table A 3, to give ϵ, and table B 6 used to obtain c_*.

Results are plotted on figure 2. The numerically exact results of Cokelet are shown by the solid lines, except that the infinite wavelength (solitary wave) results are taken from Longuet-Higgins & Fenton (1974). Alongside each curve is shown a number which is the approximate value of $\lambda/\bar{\eta}$ on that curve, which really is drawn for constant $\lambda c/Q$, a rather artificial depth scale but one which it was simpler for Cokelet to use.

Results from the present cnoidal theory are shown by the fifth and ninth order, with results from fifth-order Stokes wave theory. Clearly cnoidal wave theory gives excellent agreement for almost all wave heights up to about a maximum of 0·65, for wavelength/depth ratios of 9 or more. It is interesting that there is a sudden change between $\lambda/\bar{\eta} = 7$ and 9. For longer waves, cnoidal theory is excellent and Stokes theory not good except for low waves, while for shorter waves the situation is reversed. From this

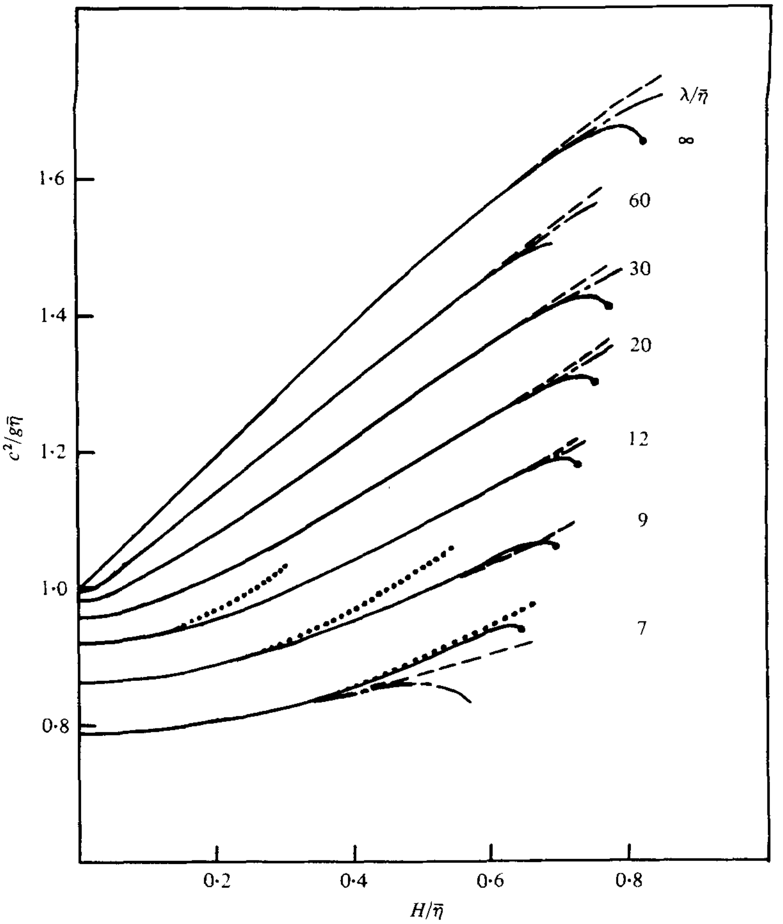


FIGURE 2. Dimensionless wave speed squared, $c^2/(g\bar{\eta})$. Comparison between the present theory (5th order ---, 9th order - - -), Stokes wave theory (5th order $\cdots$) and known results (——). Solitary wave ($\lambda = \infty$) results taken from Longuet-Higgins & Fenton (1974), all others from Cokelet (1977). The wavelength:depth ratios given are approximate.

it can be concluded that the boundary of applicability between deep and shallow water theories is at $\lambda/\eta \approx 8$.

Another result that is of interest is the apparent little gain to be had by using ninth-order results rather than fifth order. For long waves, the ninth gives slightly greater accuracy; however, for shorter waves, the effect of m decreasing in ϵ/m is felt and the ninth-order solution diverges disastrously. This behaviour is similar to that of many asymptotic series for which the inclusion of higher-order terms decreases accuracy, which suggests here that the expansion for wave speed is an asymptotic rather than a convergent series. It seems highly probable that all of the series generated in the present work are of this asymptotic form. For practical purposes it is convenient that the fifth-order solution seems to be so accurate for all long waves and that the series can be truncated at this order.

Finally, it may be noticed that the phenomenon of the speed having a maximum is not described using the present expansion in wave height. If one were to recast the expansions in other parameters, then for sufficiently high-order expansions and with

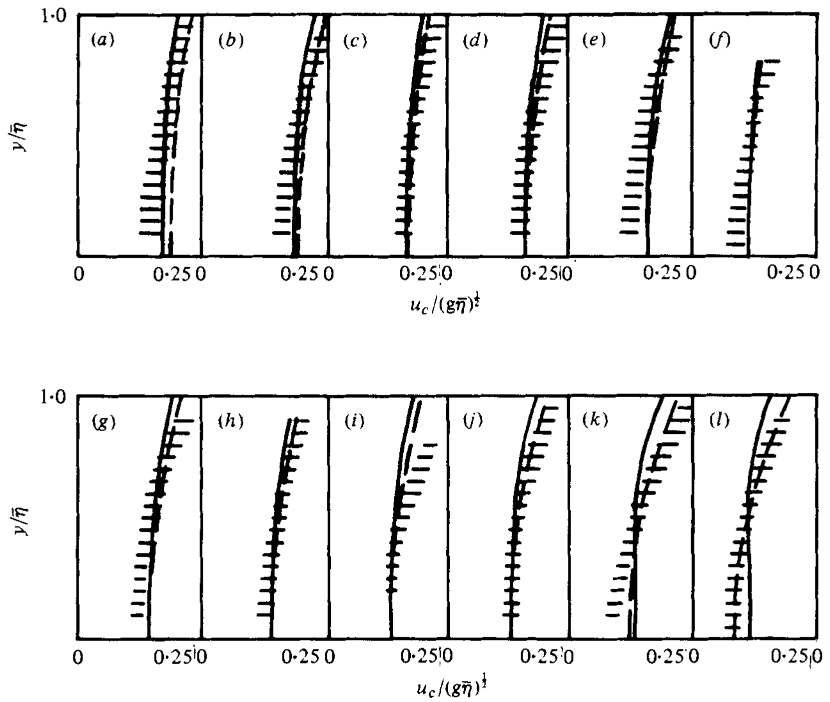


FIGURE 3. Vertical distribution of horizontal fluid velocity under wave crests. Experimental results of Iwagaki & Sakai (1970) are within the horizontally hatched regions. Fifth-order cnoidal wave theory (——), fifth-order Stokes wave theory (---). (*a*) $\bar{\epsilon} = 0{\cdot}282$, $\tau_* = 16{\cdot}8$ ($\lambda_* \approx 17{\cdot}3$); (*b*) $\bar{\epsilon} = 0{\cdot}324$, $\tau_* = 13{\cdot}6$ ($\lambda \approx 13{\cdot}9$); (*c*) $\bar{\epsilon} = 0{\cdot}297$, $\tau_* = 12{\cdot}9$ ($\lambda_* \approx 13{\cdot}0$); (*d*) $\bar{\epsilon} = 0{\cdot}306$, $\tau_* = 12{\cdot}0$ ($\lambda_* \approx 12{\cdot}0$); (*e*) $\bar{\epsilon} = 0{\cdot}308$, $\tau_* = 11{\cdot}9$ ($\lambda_* \approx 11{\cdot}9$); (*f*) $\bar{\epsilon} = 0{\cdot}239$, $\tau_* = 10{\cdot}0$ ($\lambda_* \approx 9{\cdot}6$); (*g*) $\bar{\epsilon} = 0{\cdot}318$, $\tau_* = 9{\cdot}6$ ($\lambda_* \approx 9{\cdot}3$); (*h*) $\bar{\epsilon} = 0{\cdot}307$, $\tau_* = 8{\cdot}6$ ($\lambda \approx 8{\cdot}1$); (*i*) $\bar{\epsilon} = 0{\cdot}306$, $\tau_* = 8{\cdot}3$ ($\lambda_* \approx 7{\cdot}8$); (*j*) $\bar{\epsilon} = 0{\cdot}326$, $\tau_* = 7{\cdot}4$ ($\lambda \approx 6{\cdot}8$); (*k*) $\bar{\epsilon} = 0{\cdot}345$, $\tau_* = 6{\cdot}6$ ($\lambda_* \approx 5{\cdot}9$); (*l*) $\bar{\epsilon} = 0{\cdot}298$, $\tau_* = 5{\cdot}3$ ($\lambda \approx 4{\cdot}4$).

convergence improvement techniques, exact numerical results could presumably be obtained, as did Longuet-Higgins & Fenton (1974) for the solitary wave and Cokelet (1977) for Stokes waves. However, one of the main purposes of the present work is to give a usable expansion for shallow water waves which gives quite accurate results at relatively low order, so that the duplication of previously obtained accurate numerical results seems unnecessary.

5.3. *Validity of expressions for fluid velocity*

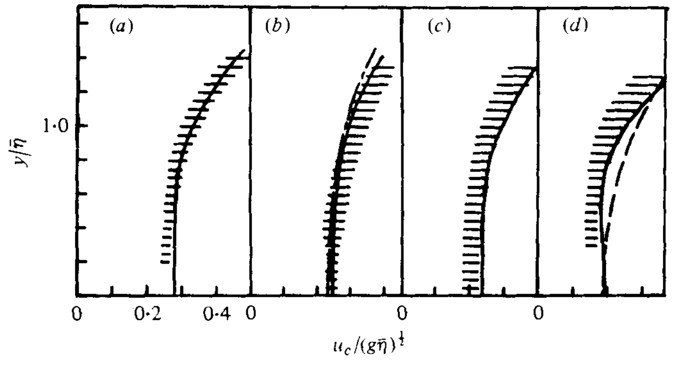
As stated in the introduction, the stimulus for this work was the need for an accurate shallow water wave theory to give the fluid flow field under waves. Two experimental investigations which measured the fluid velocity as waves passed along a laboratory tank are those reported by Le Méhauté, Divoky & Lin (1968) and Iwagaki & Sakai (1970). The most important velocity measured in each case was the horizontal velocity under the crest and its variation with depth. From each of these papers, the present author drew approximate envelope curves to all of the experimental points, which were scaled and plotted on figures 3 and 4, so that almost all of their results fell within the horizontally hatched regions. Each profile was plotted with the dimensionless horizontal velocity under the crest $u_c(y)/(g\bar{\eta})^{\frac{1}{2}}$ as abscissa and dimensionless height above the bottom $y/\bar{\eta}$ as ordinate. Every profile is identified by a reference letter, and contains two numbers: wave height $\bar{\epsilon} = H/\bar{\eta}$ and period τ_*, plus another number in brackets the wavelength/depth ratio $\lambda/\bar{\eta}$ given by fifth-order cnoidal theory. Conveniently for comparison, the wave height throughout figure 3 is about 0·3 (i.e. 0·282–0·345), while the wave height in figures 4 (*a*)–(*d*) is about 0·4, and is approximately 0·5 in 4 (*e*)–(*h*). Each figure begins with the longest wave.

On each experimental profile has been plotted one or all of the following theoretical profiles: (i) fifth-order cnoidal wave shown by a solid line, (ii) fifth-order Stokes wave, the dashed line, and (iii) high-order stream function theory (Dean 1970) shown by a chain-dotted line (on 4*b*, *e*, and *h*).

Wave height 0·3. In figures 3 (*a*)–(*e*), results from the 4th–9th cnoidal wave theories agreed so closely as to be indistinguishable, so that the author is satisfied that the lines as plotted are an accurate solution of the irrotational flow problem. Unfortunately in none of the cases do the results agree particularly well with experiment, but in view of the close agreement between the cnoidal wave theories one is tempted to believe that the disagreement between theory and experiment is due largely to real fluid effects in the experimental tank. Examining the experimental results there seems to be evidence of boundary layers near the bottom of the tank. In a full-scale situation, at Reynolds numbers some thousands of times greater, such viscous effects would be much less. Accordingly, figure 3 will be used only as a basis for comparing Stokes and cnoidal wave theory. For the longest wave, figure 3 (*a*), Stokes theory gave velocities some 15 % greater than the 4th–9th order cnoidal theories. At $\tau_* \approx \lambda/\bar{\eta} \approx 12$, figure 3 (*d*), Stokes results were within 5 % of cnoidal, corresponding to an empirical drag force (proportional to u_c^2) error of 10 %. As the wavelength decreases, the two theories agree more closely, until at $\tau_* \approx 10$ they give good agreement. From this point however, the 9th-order cnoidal theory started to be distinguishable from the 5th, and one no longer can have so much confidence in the cnoidal results. This divergence began to be marked after (*h*), so that the Stokes wave results should be trusted more for $\tau_* < 8{\cdot}6$, which is in keeping with the results of § 5.2 for wave speed. It is interesting that in figure 3 (*l*), the shortest wave, agreement between the Stokes results and the experiment was remarkably good. There is no evidence of a boundary layer, which result might be most readily expected for these shorter wavelengths.

Wave height 0·4. In figures 4 (*a*)–(*d*), for higher, but longer, waves than figure 3, agreement between the fifth-order cnoidal wave theory and experiment is good, even for wavelengths as short as 8·4 ($\tau_* = 8{\cdot}6$). However, for a real solution of the transcendental fifth-order Stokes equation for (ak), the Stokes expansion parameter could only be obtained for case (*d*). For this case, the cnoidal wave solution agrees rather better with experiment. The only stream function result given by Dean for this row is that of figure 4 (*b*), which was stated by him to be the closest to experiment of all the eight cases tested by him. It is not quite as good as the cnoidal result. An advantage of the two formal expansion procedures, Stokes and cnoidal, is that, once coefficients in the expansions are known, it is a simple matter (albeit lengthy without a computer) to produce any results given by that theory for any depth, provided the expansion is still valid. For solutions such as Dean's stream function theory, numerical solution of the equations, and hence special computer programs, are necessary.

Wave height 0·5. For higher waves still, figures 4 (*e*)–(*h*) show that the fifth-order Stokes wave solution can still only be obtained for the shortest wave and that in that



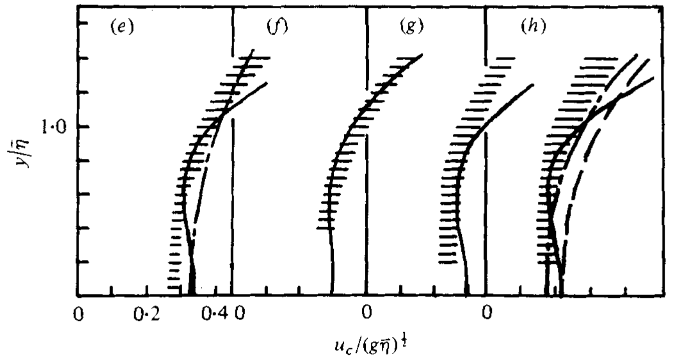


FIGURE 4. Vertical distribution of horizontal fluid velocity under wave crests. Experimental results of Le Méhauté *et al.* (1968) are within the horizontally hatched regions. Fifth-order cnoidal wave theory (——), fifth-order Stokes wave theory (– – –), stream function theory (–·–) (Dean, 1970). (*a*) $\bar{\epsilon} = 0{\cdot}433, \tau_* = 27{\cdot}2$ ($\bar{\lambda}_* \approx 30{\cdot}6$); (*b*) $\bar{\epsilon} = 0{\cdot}389, \tau_* = 22{\cdot}5$ ($\bar{\lambda}_* \approx 30{\cdot}6$); (*c*) $\bar{\epsilon} = 0{\cdot}420$, $\tau_* = 15{\cdot}9$ ($\bar{\lambda}_* \approx 16{\cdot}9$); (*d*) $\bar{\epsilon} = 0{\cdot}434, \tau_* = 8{\cdot}6$ ($\bar{\lambda}_* \approx 8{\cdot}4$); (*e*) $\bar{\epsilon} = 0{\cdot}548, \tau_* = 27{\cdot}3$ ($\bar{\lambda}_* \approx 31{\cdot}7$); (*f*) $\bar{\epsilon} = 0{\cdot}493$, $\tau_* = 22{\cdot}5$ ($\bar{\lambda}_* = 25{\cdot}3$); (*g*) $\bar{\epsilon} = 0{\cdot}522$, $\tau_* = 15{\cdot}9$ ($\bar{\lambda}_* \approx 17{\cdot}5$); (*h*) $\bar{\epsilon} = 0{\cdot}499$, $\tau_* = 8{\cdot}6$ ($\bar{\lambda}_* = 8{\cdot}5$).

case it is not accurate. The cnoidal wave results also are not good, except in the lowest case figure 4 (*f*), when $\bar{\epsilon} = 0{\cdot}492$. For the other higher waves it is no longer accurate. At this height, the stream function theory does give quite good results: not really to be preferred over cnoidal theory for the long wave case of figure 4 (*e*), but certainly for the shorter wave of figure 4 (*h*). The stream function approach should be most suited to shorter waves, as it is based on a Stokes type of expansion.

Finally, the apparently asymptotic nature of the series used for the velocity profile should be mentioned. For the series of wave height 0·3, all higher-order cnoidal expansions tended to agree with the fifth-order result, and, where small differences were apparent, the higher-order solutions would be trusted. However, for all higher waves (0·4 and 0·5), the higher-order theories gave results which were wildly divergent from the fifth-order and from the experimental results, providing more evidence for the asymptotic nature of the series. Generally the fifth-order solution was quite accurate and, except for high and short waves, did not show the marked disagreement with experiment of higher orders. In all practical application it is recommended that terms higher than the fifth be not included.

6. Conclusions

A method has been developed for generating high-order cnoidal wave solutions. Such a solution has been obtained to ninth order using a computer to manipulate the long series necessary; however, comparison with previous experimental and theoretical results has shown that there is no gain in accuracy to be had by including terms after the fifth, suggesting that the series are asymptotic rather than convergent. The fifth-order cnoidal wave solution is presented in the appendix in the form of a number of series for which all numerical coefficients are given. Any steady wave problem in shallow water, involving a given wave height, period, and water depth, may be solved using the series presented.

Results from the fifth-order theory have been compared with other high-order theories for steady water waves. The following conclusions can be drawn.

(1) For dimensionless periods $\tau(g/\bar{\eta})^{\frac{1}{2}}$, or wavelengths $\lambda/\bar{\eta}$, greater than 8, fifth-order cnoidal wave theory should be used. In the case of waves shorter than this, fifth-order Stokes wave theory is preferable.

(2) If overall integral quantities such as wave speed, wave power, etc., are required, the fifth-order cnoidal wave solution is highly accurate up to a relative wave height $H/\bar{\eta}$ of about 0·65, provided the limitation of (1) is observed.

(3) For solutions of the fluid flow beneath waves, the fifth-order cnoidal wave expansion gives accurate results for relative wave heights up to 0·4, and in the absence of other convenient theories can be applied without gross error to wave heights of 0·5.

(4) For lower wave heights there is greater overlap between the areas of validity of the two theories. For example, a wave of 0·2 times the depth may be solved using Stokes theory for relative wavelengths of up to 12 say, while cnoidal theory may be applied to wavelengths as small as 5.

Appendix

Below are set out all the fifth-order expressions based on cnoidal wave theory which have been generated in the present work. All coefficients are given rounded to five decimal places; where any fewer places are given, all trailing numbers are zero. This accuracy was considered by the author to be reasonable for all practical use, however all coefficients given are stored to ten decimal places in the computer at the University of N.S.W. The author would be glad to send punched computer cards containing these numbers to anybody who may request this.

Appendix A contains three tables of coefficients, any two of which may be used, given a practical problem involving known water depth $\bar{\eta}$, wave height H and period τ or wavelength λ, first, to solve for m by trial and error using tables A 1 or A 2, and then to solve for h the trough depth using table A 3, to give $\epsilon = H/h$, used throughout appendices B and C.

Appendix B contains tables of coefficients so that a detailed solution of the flow field for a particular wave may be generated. Finally, appendix C gives tables for several constant integral quantities of a cnoidal wave train.

Appendix A

Please note that the dimensionless wave height used throughout this appendix is $\bar{\epsilon} = H/\bar{\eta}$.

i	k	$j = 0$	1	2	3	4
1	0	1·25	−0·625	—	—	—
	1	−1·5	—	—	—	—
2	0	−0·46875	0·46875	−0·16406	—	—
	1	0·125	−0·06250	—	—	—
	2	0·375	—	—	—	—
3	0	1·01556	−1·52333	0·73241	−0·11232	—
	1	−0·91938	0·91938	−0·06391	—	—
	2	0·21875	−0·10938	—	—	—
	3	0·06250	—	—	—	—
4	0	−2·79984	5·59969	−4·07395	1·27410	−0·05493
	1	3·66490	−5·49735	3·04491	−0·60623	—
	2	−1·48453	1·48453	−0·10465	—	—
	3	0·20313	−0·10156	—	—	—
	4	0·02344	—	—	—	—

TABLE A 1. Coefficients λ_{ijk} in series for dimensionless wavelength $\lambda_* = \lambda/\bar{\eta} = 4K(3\bar{\epsilon}/m)^{-\frac{1}{2}}(1+\Sigma\Sigma\Sigma\lambda_{ijk}(\bar{\epsilon}/m)^i m^j(E/K)^k)$.

i	k	$j = 0$	1	2	3	4
1	0	0·25	−0·125		—	—
	1	—	—	—	—	—
2	0	0·01458	−0·01458	−0·07656	—	—
	1	−1·08333	0·54167	—	—	—
	2	1	—	—	—	—
3	0	0·36121	−0·54182	0·41216	−0·11578	—
	1	2·50417	−2·50417	0·33229	—	—
	2	−4·5	2·25	—	—	—
	3	2·0	—	—	—	—
4	0	−1·86885	3.73770	−2·73031	0·86147	−0·07582
	1	−4.22859	6·34288	−1·88433	−0·11498	—
	2	15·19111	−15·19111	2·69111	—	—
	3	−13·66667	6·83333	—	—	—
	4	4	—	—	—	—

TABLE A 2. Coefficients τ_{ijk} in series for dimensionless period $\tau_* = \tau(g/\bar{\eta})^{\frac{1}{2}} = 4K(3\bar{\epsilon}/m)^{-\frac{1}{2}}(1+\Sigma\Sigma\Sigma\tau_{ijk}(\bar{\epsilon}/m)^i m^j(E/K)^k)$.

i	k	$j = 0$	1	2	3	4
1	0	1·0	−1·0	—	—	—
	1	−1	—	—	—	—
2	0	−0·5	0·5	—	—	—
	1	0·5	−0·25	—	—	—
	2	—	—	—	—	—
3	0	0·665	−0·99750	0·33250	—	—
	1	−1·165	1·165	−0·04	—	—
	2	0·5	−0·25	—	—	—
	3	—	—	—	—	—
4	0	−1·62667	3·25333	−2·454	0·82733	—
	1	*3·20667*	*−4·81*	*2·17633*	*−0·28650*	—
	2	*−2·08*	*2·08*	*−0·14250*	—	—
	3	0·5	−0·25	—	—	—
	4	—	—	—	—	—
5	0	4·86659	−12·16647	11·79929	−5·53247	1·03306
	1	−10·74409	21·48818	−16·00776	5·26368	−0·20555
	2	8·62250	−12·93375	6·09025	−0·88950	—
	3	−3·245	3·245	−0·30750	—	—
	4	0·5	−0·25	—	—	—
	5	—	—	—	—	—

TABLE A 3. Coefficients h_{ijk} in series for minimum depth
$h/\bar{\eta} = 1 + \Sigma\Sigma\Sigma h_{ijk}(\bar{\epsilon}/m)^i m^j (E/K)^k$.

Appendix B

The dimensionless wave height used throughout this appendix is $\epsilon = H/h$.

i	$j = 0$	1	2	3	4
1	0·25	−0·875	—	—	—
2	0·03125	−0·34375	0·86719	—	—
3	−0·37743	0·51146	0·13743	−0·833	—
4	0·20322	0·44278	−1·38945	0·54282	0·76773

TABLE B 1. Coefficients α_{ij} in series for α: $\alpha = (\frac{3}{4}\epsilon/m)^{\frac{1}{2}}(1 + \Sigma\Sigma\alpha_{ij}(\epsilon/m)^i m^j)$.

i	j	k = 1	2	3	4	5
1	0	—	—	—	—	—
	1	1·0	—	—	—	—
2	0	—	—	—	—	—
	1	—	—	—	—	—
	2	−0·75	0·75	—	—	—
3	0	—	—	—	—	—
	1	—	—	—	—	—
	2	−0·76250	0·76250	—	—	—
	3	1·38750	−2·65	1·26250	—	—
4	0	—	—	—	—	—
	1	—	—	—	—	—
	2	−0·80533	0·80533	—	—	—
	3	2·48904	−4·33146	1·84242	—	—
	4	−3·05188	7·40646	−6·52546	2·17088	—
5	0	—	—	—	—	—
	1	—	—	—	—	—
	2	0·43643	−0·43643	—	—	—
	3	1·92280	−4·66167	2·73888	—	—
	4	−7·04588	17·45561	−15·31697	4·90723	—
	5	6·54722	−19·80887	25·34187	−16·32709	4·24687

TABLE B 2. Coefficients η_{ijk} in series for wave profile
$\eta = 1+\Sigma\Sigma\Sigma\eta_{ijk}(\epsilon/m)^i m^j(\mathrm{cn}^2)^k$.

i	$j = 0$	1	2	3	4	5
1	−0·5	1·0	—	—	—	—
2	0·225	−0·35	−0·025	—	—	—
3	−0·07857	0·06161	0·04911	0·02143	—	—
4	0·39788	−0·73683	0·70620	−0·38355	−0·03888	—
5	−0·82992	1·57745	−0·59145	−0·67125	0·46271	0·07232

TABLE B 3. Coefficients Q_{ij} in series for dimensionless volume flux
$Q_* = 1+\Sigma\Sigma Q_{ij}(\epsilon/m)^i m^j$.

i	$j = 0$	1	2	3	4	5
1	−0·5	1·0	—	—	—	—
2	0·35	−0·35	−0·025	—	—	—
3	−0·19107	0·11161	0·01161	0·04643	—	—
4	0·46248	−0·73388	0·65870	−0·31944	−0·085	—
5	−1·04654	1·92262	−0·98229	−0·39222	0·32619	0·15647

TABLE B 4. Coefficients R_{ij} in series for dimensionless energy per unit mass
$R_* = 1{\cdot}5+\Sigma\Sigma R_{ij}(\epsilon/m)^i m^j$.

i	l	j	$k = 0$	1	2	3	4	5
1	0	0	−0·5	—	—	—	—	—
		1	1·0	−1·0	—	—	—	—
2	0	0	0·225	—	—	—	—	—
		1	−0·6	1·0	—	—	—	—
		2	0·225	−1·25	1·0	—	—	—
	1	0	—	—	—	—	—	—
		1	0·75	−1·5	—	—	—	—
		2	−0·75	3·0	−2·25	—	—	—
3	0	0	−0·07857	—	—	—	—	—
		1	0·14911	−0·4	—	—	—	—
		2	0·16161	1·35	−1·7	—	—	—
		3	−0·17857	−0·475	1·9	−1·2	—	—
	1	0	—	—	—	—	—	—
		1	−0·375	0·75	—	—	—	—
		2	—	−5·25	7·125	—	—	—
		3	0·375	3·0	−10·875	7·5	—	—
	2	0	—	—	—	—	—	—
		1	0·18750	−0·375	—	—	—	—
		2	−0·56250	3·18750	−2·81250	—	—	—
		3	0·375	−3·18750	5·625	−2·81250	—	—
4	0	0	0·39788	—	—	—	—	—
		1	−0·74576	0·09643	—	—	—	—
		2	0·73477	−0·46431	0·81967	—	—	—
		3	−0·67908	0·55694	−3·04667	2·90133	—	—
		4	0·23701	0·17018	1·03417	−3·10267	1·576	—
	1	0	—	—	—	—	—	—
		1	0·01875	−0·03750	—	—	—	—
		2	−0·09375	5·625	−7·25625	—	—	—
		3	0·66562	−7·55625	29·025	−25·2	—	—
		4	−0·59063	3·03750	−17·07188	33·525	−18·9	—
	2	0	—	—	—	—	—	—
		1	—	—	—	—	—	—
		2	0·14063	−4·78125	6·0	—	—	—
		3	0·14063	9·70312	−33·42187	24·375	—	—
		4	−0·28125	−4·07812	27·09375	−42·42187	19·68750	—
	3	0	—	—	—	—	—	—
		1	0·01875	−0·03750	—	—	—	—
		2	−0·17813	1·23750	−1·18125	—	—	—
		3	0·31875	−3·48750	7·08750	−3·93750	—	—
		4	−0·15938	2·325	−7·08750	7·875	−2·95313	—
5	0	0	−0·82992	—	—	—	—	—
		1	1·76991	−0·78281	—	—	—	—
		2	−0·99629	0·89794	0·49893	—	—	—
		3	−0·41272	−0·66324	1·20740	−2·26014	—	—
		4	0·83138	0·70429	−2·71917	7·12267	−4·96993	—
		5	−0·34250	−0·26144	0·43733	−2·69367	5·21186	−2·193
	1	0	—	—	—	—	—	—
		1	−0·57050	1·141	—	—	—	—
		2	1·44569	−4·45837	2·27950	—	—	—
		3	−1·29006	5·86650	−31·13156	31·778	—	—
		4	−0·38012	−2·12075	42·62962	−110·618	74·07	—
		5	0·795	−2·23875	−11·405	61·73675	−91·44	42·552
	2	0	—	—	—	—	—	—
		1	−0·01875	0·03750	—	—	—	—
		2	−0·45	3·84375	−3·91875	—	—	—
		3	0·96797	−16·09219	63·225	−53·23125	—	—
		4	−1·16016	16·30547	−110·93203	225·63750	−131·90625	—
		5	0·66094	−5·39297	47·56406	−162·53203	204·75	−85·05

i	l	j	k = 0	1	2	3	4	5
	3	0	—	—	—	—	—	—
		1	0·00937	−0·01875	—	—	—	—
		2	0·11719	−1·59375	1·80938	—	—	—
		3	−0·25313	9·84375	−31·55625	22·96875	—	—
		4	0·05156	−13·23750	73·18125	−115·5	55·61719	—
		5	0·075	4·875	−41·7	103·68750	−102·375	35·43750
	4	0	—	—	—	—	—	—
		1	0·001	−0·00201	—	—	—	—
		2	−0·03415	0·26016	−0·25614	—	—	
		3	0·12656	−1·64833	3·70647	−2·21484	—	—
		4	−0·15569	2·77634	−9·58259	11·70703	−4·74609	—
		5	0·06228	−1·38817	6·38839	−11·70703	9·49219	−2·84766

TABLE B 5. Coefficients ϕ_{ijkl} in series for fluid velocities, accelerations and pressure.

i	k	j = 0	1	2	3	4	5
1	0	0·5	—	—	—	—	—
	1	−1·0	—	—	—	—	—
2	0	−0·10833	−0·01667	−0·025	—	—	—
	1	0·33333	0·08333	—	—	—	—
3	0	−0·17190	0·33911	−0·16006	0·04643	—	—
	1	0·09333	−0·34333	0·21833	—	—	—
4	0	0·02097	0·17293	−0·56238	0·39861	−0·08531	—
	1	0·37690	−0·68202	1·04889	−0·56668	—	—
5	0	0·11046	−0·31285	−0·11262	0·91605	−0·73881	0·15763
	1	−0·94038	1·22117	0·35314	−1·75325	1·00619	—

TABLE B 6. Coefficients c_{ijk} in series for wave speed
$c_* = 1+\Sigma\Sigma\Sigma c_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

Appendix C

The dimensionless wave height used throughout this appendix is $\epsilon = H/h$.

i	k	$j = 0$	1	2	3	4
2	0	−0·33333	0·33333	—	—	—
	1	1·33333	−0·66667	—	—	—
	2	−1·0	—	—	—	—
3	0	0·1	−0·06667	−0·03333	—	—
	1	−0·93333	0·6	0·06667	—	—
	2	0·83333	−0·16667	—	—	—
4	0	0·205	−0·485	0·35417	−0·07417	—
	1	0·03333	0·65	−0·83	0·14833	—
	2	−0·23833	−0·38667	0·44917	—	—
5	0	−0·10943	0·01732	0·56278	−0·67712	0·20644
	1	−0·82248	1·60329	−2·67396	2·18065	−0·41289
	2	0·93190	−1·09994	1·64231	−0·98610	—

TABLE C 1. Coefficients I_{ijk} in series for wave impulse $I_* = \Sigma\Sigma\Sigma I_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	k	$j = 0$	1	2	3	4
2	0	−0·16667	0·16667	—	—	—
	1	0·66667	−0·33333	—	—	—
	2	−0·5	—	—	—	—
	3	—	—	—	—	—
3	0	−0·03333	0·05	−0·01667	—	—
	1	0·03333	−0·03333	0·03333	—	—
	2	−0·5	0·25	—	—	—
	3	0·5	—	—	—	—
4	0	0·14556	−0·27444	0·17014	−0·04125	—
	1	−0·39444	0·575	−0·37889	0·08250	—
	2	0·83222	−0·58222	0·17597	—	—
	3	−0·58333	0·04167	—	—	—
5	0	0·01977	−0·19498	0·45424	−0·39040	0·11138
	1	−0·56834	1·53113	−2·04019	1·27574	−0·22275
	2	0·33718	−1·338	1·71556	−0·65835	—
	3	0·21139	0·37194	−0·34069	—	—

TABLE C 2. Coefficients T_{ijk} in series for kinetic energy $T_* = \Sigma\Sigma\Sigma T_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	k	$j = 0$	1	2	3	4
2	0	−0·16667	0·16667	—	—	—
	1	0·66667	−0·33333	—	—	—
	2	−0·5	—	—	—	—
3	0	0·1	−0·15	0·05	—	—
	1	−0·6	0·6	−0·1	—	—
	2	0·5	−0·25	—	—	—
4	0	0·00905	−0·06810	0·10744	−0·04839	—
	1	0·28095	−0·12143	−0·25310	0·09679	—
	2	−0·29	0·04	0·17875	—	—
5	0	0·03644	−0·10014	0·24178	−0·27687	0·09880
	1	−0·66560	1·05025	−1·21573	0·92465	−0·19760
	2	0·62917	−0·65375	0·72508	−0·404	—

TABLE C 3. Coefficients V_{ijk} in series for potential energy $V_* = \Sigma\Sigma\Sigma V_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	k	$j = 0$	1	2	3	4
2	0	−0·33333	0·33333	—	—	—
	1	1·33333	−0·66667	—	—	—
	2	−1·0	—	—	—	—
3	0	0·4	−0·43333	0·03333	—	—
	1	−1·06667	0·4	−0·06667	—	—
	2	0·66667	0·16667	—	—	—
4	0	−0·05016	−0·29968	0·50853	−0·15869	—
	1	−0·02540	1·10476	−0·91413	0·31738	—
	2	0·07556	−0·74222	0·42972	—	—
5	0	0·09597	−0·16468	0·61066	−0·84791	0·30597
	1	−0·78756	1·61321	−3·224	2·48383	−0·61194
	2	0·69159	−1·15071	2·00945	−1·16975	—

TABLE C 4. Coefficients U_{ijk} in series for mean square of bed velocity $\overline{u_b^2}/gh = \Sigma\Sigma\Sigma U_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	k	$j = 0$	1	2	3	4
2	0	−0·5	0·5	—	—	—
	1	2·0	−1·0	—	—	—
	2	−1·5	—	—	—	—
	3	—	—	—	—	—
3	0	0·3	−0·45	0·15	—	—
	1	−0·8	0·8	−0·3	—	—
	2	−0·5	0·25	—	—	—
	3	1·0	—	—	—	—
4	0	−0·06175	−0·02651	0·23343	−0·14518	—
	1	−0·14603	0·61905	−0·75373	0·29036	—
	2	1·37444	−1·12444	0·53069	—	—
	3	−1·16667	0·08333	—	—	—
5	0	0·37090	−0·86550	1·07073	−0·87252	0·29639
	1	−2·09721	4·34044	−5·24517	3·28264	−0·59279
	2	1·30353	−3·32974	4·15620	−1·72069	—
	3	0·42278	0·74389	−0·68139	—	—

TABLE C 5. Coefficients $(S_{xx})_{ijk}$ in series for radiation stress $S^*_{xx} = \Sigma\Sigma(S_{xx})_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	k	$j = 0$	1	2	3	4
2	0	−0·33333	0·33333	—	—	—
	1	1·33333	−0·66667	—	—	—
	2	−1·0	—	—	—	—
	3	—	—	—	—	—
	4	—	—	—	—	—
3	0	−0·1	0·06667	0·03333	—	—
	1	0·93333	−0·6	−0·06667	—	—
	2	−2·83333	1·16667	—	—	—
	3	2·0	—	—	—	—
	4	—	—	—	—	—
4	0	0·23516	−0·44532	0·30813	−0·09798	—
	1	−1·40794	1·87857	−0·91254	0·19595	—
	2	3·00611	−2·29778	0·61028	—	—
	3	−1·33333	0·16667	—	—	—
	4	−0·5	—	—	—	—
5	0	0·20883	−0·69843	1·06473	−0·79994	0·22482
	1	−1·26248	3·58090	−4·88547	2·87763	−0·44963
	2	0·66365	−3·25006	4·58302	−1·70602	—
	3	−0·36	1·36	−1·235	—	—
	4	0·75	—	—	—	—

TABLE C 6. Coefficients F_{ijk} in series for wave power
$F_* = \Sigma\Sigma\Sigma F_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	k	$j = 0$	1	2	3	4
2	0	0·33333	−0·33333	—	—	—
	1	−1·33333	0·66667	—	—	—
3	0	0·23333	−0·6	0·36667	—	—
	1	−0·73333	1·733333	−0·73333	—	—
4	0	−0·13833	−0·015	0·44583	−0·29250	—
	1	−0·16667	1·16667	−1·58667	0·585	—
5	0	−0·09057	0·46851	−0·64195	0·17795	0·08606
	1	1·00248	−1·71329	1·36479	−0·35648	−0·17211

TABLE C 7. Coefficients C_{ijk} in series for mean Stokes drift velocity
$C_s^* = \Sigma\Sigma\Sigma C_{ijk}(\epsilon/m)^i m^j (E/K)^k$.

i	$j = 0$	1	2	3	4	5
1	−1·0	2·0	—	—	—	—
2	0·7	−1·2	0·45	—	—	—
3	−0·38214	0·57321	−0·27679	0·04286	—	—
4	0·92495	−1·65883	1·48258	−0·70585	−0·07714	—
5	−2·09307	4·30772	−2·68650	−0·19726	0·49464	0·14295

TABLE C 8. Coefficients S_{ij} in series for momentum flux
$S = 1\cdot5 + \Sigma\Sigma(\epsilon/m)^i m^j S_{ij}$.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions.* New York: Dover.

BENJAMIN, T. B. & LIGHTHILL, M. J. 1954 On cnoidal waves and bores. *Proc. R. Soc.* A **224**, 448–460.

COKELET, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc.* A **286**, 183–230.

DE, S. C. 1955 Contributions to the theory of Stokes waves. *Proc. Camb. Phil. Soc.* **51**, 713–736.

DEAN, R. C. 1970 Recent results obtained from a numerical wave theory for highly nonlinear water waves. *Proc. Symp. on Long Waves*, pp. 129–152. University of Delaware, Newark, Delaware.

FENTON, J. D. 1972 A ninth-order solution for the solitary wave. *J. Fluid Mech.* **53**, 257–271.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1965 *Table of Integrals, Series and Products.* New York: Academic.

IWAGAKI, Y. & SAKAI, T. 1970 Horizontal water particle velocity of finite amplitude waves. *Proc. 12th Conf. Coastal Engng.* **1**, 309–325.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave. *Phil. Mag.* (5) **39**, 422–443.

LAITONE, E. V. 1960 The second approximation to cnoidal and solitary waves. *J. Fluid Mech.* **9**, 430–444.

LAITONE, E. V. 1965 Series solutions for shallow water waves. *J. Geophys. Res.* **70**, 995–998.

LE MÉHAUTÉ, B., DIVOKY, D. & LIN, A. 1968 Shallow water waves: a comparison of theories and experiments. *Proc. 11th Conf. Coastal Engng.* **1**, 86–107.

LIGHTHILL, M. J. 1949 A technique for rendering approximate solutions to physical problems uniformly valid. *Phil. Mag.* **40**, 1179–1201.

LITTMAN, W. 1957 On the existence of periodic waves near critical speed. *Commun. Pure Appl. Math.* **10**, 241–269.

LONGUET-HIGGINS, M. S. 1975 Integral properties of periodic gravity waves of finite amplitude. *Proc. R. Soc.* A **342**, 157–174.

LONGUET-HIGGINS, M. S. & FENTON, J. D. 1974 On the mass, momentum, energy and circulation of a solitary wave. II. *Proc. R. Soc.* A **340**, 471–493.

MONKMEYER, P. L. 1970 A higher order theory for symmetrical gravity waves. *Proc. 12th Conf. Coastal Engng.* **1**, 543–561.

SCHWARTZ, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

SKJELBREIA, L. & HENDRICKSON, J. 1961 Fifth order gravity wave theory. *Proc. 7th Conf. Coastal Engng*, pp. 184–196.